Numerical Solutions for PDEs

We consider the diffusion equation $u_t = u_{xx}$ for $x \in [0,1]$ and $t > 0$ with initial condition $u(x,0) = f(x)$ and boundary conditions $u(0,t) = a(t)$ and $u(1,t) = b(t)$. We consider the FTCS scheme

$$U_{j,n+1} = rU_{j+1,n} + (1-2r)U_{j,n} + rU_{j-1,n}.$$

Let

$$\mathbf{U}_{0:N}^{n+1} = (U_{0,n+1}, \dots, U_{N,n+1}),$$

Find the square matrix $A \in \mathbb{M}_{N+1}(\mathbb{R})$ such that we can write

$$\mathbf{U}_{0:N}^{n+1} = A\mathbf{U}_{0:N}^{n}\ , \forall n > 0.$$

Now find the matrix $B \in \mathbb{M}_{N-1}(\mathbb{R})$ and the vector $c \in \mathbb{R}^{N-1}$ such that

$$\mathbf{U}_{1:N-1}^{n+1} = B\mathbf{U}_{1:N-1}^{n} + c\ , \forall n > 0.$$

Solution

For $n > 0$ we have

$$\begin{aligned} U_{0,n+1} &= a(t_n) \\ U_{j,n+1} &= rU_{j+1,n} + (1-2r)U_{j,n} + rU_{j-1,n}\ ,\ j = 1, \dots, N-1 \\ U_{N,n+1} &= b(t_n) \end{aligned}$$

Writing the second equation as

$$U_{j,n+1} = \begin{pmatrix} r & 1-2r & r \end{pmatrix} \begin{pmatrix} U_{j-1,n} & U_{j,n} & U_{j+1,n} \end{pmatrix} \tag{1}$$

from this we see that A can be written as

$$A = \begin{pmatrix} a(t_n) & 0 & \dots & & & & \\ r & 1-2r & r & 0 & \dots & & \\ 0 & r & 1-2r & r & 0 & \dots & \\ & & \ddots & \ddots & \ddots & & \\ & & & & & & \\ & & \dots & 0 & r & 1-2r & r \\ & & & & \dots & 0 & b(t_n) \end{pmatrix}$$

Now removing the boundary conditions from this matrix we obtain

$$B = \begin{pmatrix} 1-2r & r & 0 & \dots & & \\ r & 1-2r & r & 0 & \dots & \\ & \ddots & \ddots & \ddots & & \\ & & & & & \\ & \dots & 0 & r & 1-2r & \end{pmatrix}$$

and $c = (rU_{0,n}, 0, \dots, 0, rU_{N,n})$.

We now consider the BTCS scheme given by

$$\frac{U_{j,n} - U_{j,n-1}}{\Delta t} = \frac{U_{j+1,n} - 2U_{j,n} + U_{j-1,n}}{\Delta x^{2}}.$$

Considering the same boundary conditions find a matrix equation of the form

$$A\mathbf{U}_{0:N}^{n+1} = \mathbf{U}_{0:N}^{n} \,, \forall n > 0.$$

Find a matrix equation of the form

$$\mathbf{U}_{1:N-1}^{n+1} = B\mathbf{U}_{1:N-1}^{n} + c \,, \forall n > 0.$$