Numerical Solutions for PDEs: Lab 4

We consider the Poisson equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x, y), \quad (x, y) \in [0, 1]^2, \tag{1}$$

together with boundary conditions at $u(0, y) = u(1, y) = u(x, 0) = u(x, 1) = 0$.

We choose $(N, K) \in \mathbb{N}^2$ and we discretize the domain as follows

$$x_j = j\Delta x, \quad j = 0, \ldots, N, \tag{2}$$

$$y_k = k\Delta y, \quad k = 0, \ldots, K, \tag{3}$$

where $\Delta x = 1/N$, $\Delta y = 1/K$. We use 2nd order central difference operator for the space derivatives, we have

$$\frac{\delta_x^2 U_{j,k}}{\Delta x^{\,2}} + \frac{\delta_y^2 U_{j,k}}{\Delta y^{\,2}} = f(x_j, y_k), \tag{4}$$

for all interior points, that is

$$\frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k}}{\Delta x^{\,2}} + \frac{U_{j,k+1} - 2U_{j,k} + U_{j,k+1}}{\Delta y^{\,2}} = f(x_j, y_k). \tag{5}$$

Considering for simplicity that $\Delta x^{\,2} = \Delta y^{\,2}$, ie that $K = N$, the above equation can be rewritten as

$$U_{j+1,k} + U_{j-1,k} - 4U_{j,k} + U_{j,k+1} + U_{j,k-1} = \Delta x^{\,2} f(x_j, y_k).$$

For Dirichlet boundary conditions, we thus have a linear system in $(N-1)^2$ unknowns.

Write the scheme in matrix form and write a matlab script to find the aproximate solution with

$$f(x, y) = \exp\left(-(x - 0.5)^2 - (y - 0.5)^2\right).$$