Numerical Solutions for PDEs: Exercise sheet 5

Exercise 1:
Let u be a smooth function of x and y. Using a Taylor series expansion of $u(x+h,y)$ and $u(x,y+h)$ show that

$$\frac{\delta_x^2 u}{h^2} + \frac{\delta_y^2 u}{h^2} = u_{xx} + u_{yy} + \frac{h^2}{12}(u_{4x} + u_{4y}) + \dots,$$

then show that the finite difference scheme

$$U_{j+1,k} + U_{j-1,k} - 4U_{j,k} + U_{j,k+1} + U_{j,k-1} = \Delta x^2 f(x_j, y_k), \tag{1}$$

is a consistent approximation of the PDE $u_{xx} + u_{yy} = f(x,y)$.

Exercise 2:
The five points approximation described above is the simplest approximation one can get for $u_{xx} + u_{yy}$, it can be written as

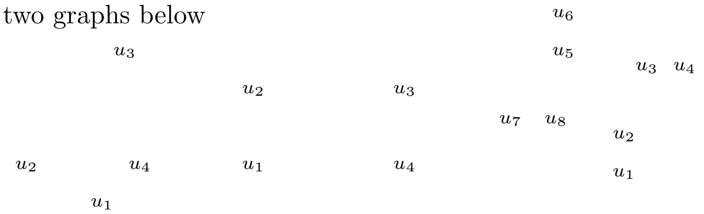
$$\alpha_1 U_1 + \alpha_2 U_2 - \alpha_0 U_0 + \alpha_3 U_3 + \alpha_4 U_4 \approx u_{xx}\Big|_{j,k} + u_{yy}\Big|_{j,k}$$

with $\alpha_0 = 4$ and $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1$ and $U_{j,k} = U_0$, $U_{j+1,k} = U_1$, $U_{j,k+1} = U_2$, $U_{j-1,k} = U_3$, $U_{j,k-1} = U_4$.

Using the Taylor series expansion of $u(x+rh,y)$ and $u(x,y+sh)$ at order 5, find the values for α_0, α_1, α_2, α_3 and α_4 such that

$$u_{xx}\Big|_{j,k} + u_{yy}\Big|_{j,k} \approx \alpha_1 U_1 + \alpha_2 U_2 - \alpha_0 U_0 + \alpha_3 U_3 + \alpha_4 U_4,$$

where $U_0 = U_{j,k}$ and u_1, u_2,u_3 and u_4 are as described on the first two graphs below, then using the Taylor series expansion of order 9 write a nine points approximation $u_{xx} + u_{yy}$ with the points configurations described on the last two graphs below



Exercise 3:
Consider Laplace equation $u_{xx} + u_{yy} = 0$ on $(0,1)^2$ together with the boundary conditions $u(0,y) = a_1$, $u(x,0) = b_1$, $u(1,y) = a_2$ and $u(x,1) = b_2$. Consider a regular grid with grid spacing $N_x = N_y = 3$, discretize the PDE using the scheme 1 and write the scheme in matrix form, finally find an approximate solution.