Numerical Solutions for PDEs: Exercise sheet 1

Exercise 1:
Let $u(x,y)$ be a solution of the equation $u_x + u_y = 0$. Use Taylor expansion to show that

$$\frac{u(x+\Delta x, y) - u(x,y)}{\Delta x} + \frac{u(x,y) - u(x, y-\Delta y)}{\Delta y} = O(\Delta x, \Delta y).$$

Exercise 2:
Let $u(x,t)$ be a solution of the equation $u_t + u_{xx} = 0$. Use Taylor expansion to show that

$$\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} + \frac{u(x+\Delta x, t) - 2u(x,t) + u(x-\Delta x, t)}{\Delta x^{\,2}} = O(\Delta x, \Delta t)$$

Exercise 3:
Let u be a function of x and t. Recall the definition of the forward operator $F_x u$, the backward operator $B_t u$, the 1st order central operator $D_x u$ and the 2nd order central operator $\delta_x^2 u$, and show that

1. $F_x B_t = B_t F_x$,
2. $F_x F_x U_{j,n} = \delta_x^2 U_{j+1,n}$, where $U_{j,n} = U(x_j, t_n)$ for a discretization $\{x_j, t_n\}$ with $x_{j+1} - x_j = t_{n+1} - t_n = \Delta x = \Delta t$, $\forall j, n$.
3. $F_t + B_t = D_t$.

Exercise 4:
Let u be a function of x, and let $\{x_j\}_{j\in\mathbb{N}}$ be a spatial discretization such that $x_{j+1} - x_j = \Delta x$. Show that

$$u_{j+1} = \exp\left(\Delta x \frac{\partial}{\partial x}\right) u_j.$$

Consider the shift operator, S, defined by $Su_j = u_{j+1}$, show that

$$S = 1 + F_x.$$

Finally show that

1. $u_x\Big|_j \Delta_x = \left[F_x - \frac{F_x^2}{2} + \frac{F_x^3}{3} - ...\right] u_j,$
2. $u_x\Big|_j \Delta_x = \left[B_x + \frac{B_x^2}{2} + \frac{B_x^3}{3} + ...\right] u_j.$