Numerical Solutions for PDEs: Exercise sheet 2

Exercise 1:
Let $u(x,t)$ be solution of the following boundary value problem

$$\begin{cases} u_t = u_{xx}, \quad 0 \le x \le 1 & , \quad u(0,t) = 0, \\ u(x,0) = 2x - x^2 & , \quad u(1,t) = 1 - t. \end{cases}$$

Write a FTCS scheme for the PDEs. Divide uniformely the spatial domain $[0,1]$ into $N = 3$ intervals, set the mesh ratio to 0.45 and perform two timesteps.

Exercise 2:
Find the LTE for the following approximations of the heat equation $u_t = u_{xx}$: 1. implicit FTCS scheme, 2. CTCS scheme.

Exercise 3:
Consider the following parabolic PDE $u_x + u_y - u_{xx} = 0$ for $u(x,y)$. Let us use the following approximation

$$u_x\Big|_{i,j} \approx \frac{D_x u_{i,j}}{\Delta x}\ , \ u_y\Big|_{i,j} \approx \frac{F_y u_{i,j}}{\Delta y},$$

$$u_{xx}\Big|_{i,j} \approx \frac{1}{3}\left[\frac{\delta_x^2 u_{i+1,j}}{\Delta y^2} + \frac{\delta_x^2 u_{i,j}}{\Delta y^2} + \frac{\delta_x^2 u_{i-1,j}}{\Delta y^2}\right],$$

where $U_{i,j} = U(x_i, x_j)$ is an approximation of u. Show that the resulting scheme is consistent with the PDE.

Exercise 4:
Consider the following approximation of the Black-Scholes equation

$$\frac{V_{j,n} - V_{j,n-1}}{\Delta t} = -\rho S_j \frac{D_S V_{j,n}}{2\Delta S} - \frac{\sigma^2}{2} S_j^2 \frac{\delta_S^2 V_{j,n}}{\Delta S^2} + \rho V_{j,n}.$$

Work out the LTE for this approximation $S_j = j\Delta S$ and $V_{j,n} \approx V(S_j, t_n)$.

Exercise 5:
Under what restrictions the following schemes are Von Neumann stable:

1. $U_{j,n+1} - U_{j,n} = r\delta_x^2\left(U_{j,n} + U_{j,n+1}\right)/2$, where U is an approximation for $u(x,t)$ solution of $u_t = u_{xx}$,
2. $U_{j,n+1} - U_{j,n} = r\delta_x^2 U_{j,n} - \lambda \Delta t\, U_{j,n+1}$, where U is an approximation for $u(x,t)$ solution of $u_t = u_{xx} - \lambda u$, $\lambda > 0$,
3. $U_{j,n+1} - U_{j,n-1} = 2r\delta_x^2 U_{j,n}$, where U is an approximation for $u(x,t)$ solution of $u_t = u_{xx}$.

$$\begin{aligned} &e^{i\omega} = \cos\omega + i\sin\omega && \cos 2\omega = \cos^2\omega - \sin^2\omega \\ &\cos^2\omega + \sin^2\omega = 1 && = 2\cos^2\omega - 1 \\ &\sin 2\omega = 2\sin\omega\cos\omega && = 1 - 2\sin^2\omega \end{aligned}$$