Numerical Solutions for PDEs
Note on the 1st coursework

Exercise 2:
Under what restriction the following scheme is Von Neumann stable:

$$U_{j,n+1} - U_{j,n-1} = 2r\delta_x^2 U_{j,n},$$

where U is an approximation for $u(x,t)$ solution of $u_t = u_{xx}$.

solution
Here we are interested with $r = \Delta t/\Delta x > 0$ which simplifies the stability analysis. Let us write the scheme as

$$U_{j,n+1} - U_{j,n-1} = 2r(U_{j+1,n} - 2U_{j+1,n} + U_{j+1,n}),$$

Inserting a trial solution of the form $U_{j,n} = A\xi^n \mathrm{e}^{\mathrm{i}\omega j}$ and dividing by $A\xi^n \mathrm{e}^{\mathrm{i}\omega j}$ gives

$$\xi^2 - 1 = 2r(\mathrm{e}^{\mathrm{i}\omega} - 2 + \mathrm{e}^{-\mathrm{i}\omega}).$$

Using the formula $\cos\omega = 1 - 2\sin^2(\omega/2)$ we obtain the characteristic equation

$$\xi^2 + 8r\sin^2\frac{\omega}{2} - 1 = 0.$$

This quadratic has two real roots given by

$$\xi_\pm = -4r\sin^2\frac{\omega}{2} \pm \sqrt{1 + 16r^2\sin^4\frac{\omega}{2}}.$$

We consider ξ_-. Since $r > 0$ we have

$$-4r\sin^2\frac{\omega}{2} - \sqrt{1 + 16r^2\sin^4\frac{\omega}{2}} \le -1$$

therefore the scheme is unstable.

Exercise 3:
The backward time, central space scheme for the diffusion equation is given by is

$$\frac{U_{j,n} - U_{j,n-1}}{\Delta t} = \frac{U_{j+1,n} - 2U_{j,n} + U_{j-1,n}}{\Delta x^2}.$$

Show that the scheme consistent. Find under what restriction on $r = \Delta t/\Delta x^2$ the scheme converges.

solution
We study the local trunctation error. We have

$$\begin{aligned} B_t u &= u(x,t) - u(x, t-\Delta t) \\ &= u - u + u_t\Delta t - u_{tt}\frac{\Delta t^2}{2} + O(\Delta t^3) \end{aligned}$$

and

$$
\begin{aligned}
\delta_x^2 &= u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t) \\
&= u + u_x\Delta x + u_{xx}\frac{\Delta x^2}{2} + O(\Delta x^3) - 2u + u - u_x\Delta x + u_{xx}\frac{\Delta x^2}{2} + O(\Delta x^3) \\
&= u_{xx}\Delta x^2 + O(\Delta x^4)
\end{aligned}
$$

Then the LTE is given

$$
\begin{aligned}
\frac{B_t u}{\Delta t} - r\frac{\delta_x^2 u}{\Delta x^2} &= u_t - u_{tt}\frac{\Delta t}{2} + O(\Delta t^2) - r u_{xx} + O(\Delta x^2) \\
&= \underbrace{u_t - r u_{xx}}_{=0} - \underbrace{u_{tt}\frac{\Delta t}{2} + O(\Delta t^2, \Delta x^2)}_{\to 0 \text{ as } \Delta t, \Delta x \to 0}
\end{aligned}
$$

therefore the scheme is consistent with the PDE. We now study the stability of the scheme. We write the scheme as

$$
U_{j,n} - U_{j,n-1} = r(U_{j+1,n} - 2U_{j,n} + U_{j-1,n})
$$

Inserting a trial solution of the form $U_{j,n} = A\xi^n \mathrm{e}^{\mathrm{i}\omega j}$ and dividing by $A\xi^{n-1}\mathrm{e}^{\mathrm{i}\omega j}$ we get

$$
\xi - 1 = r\xi(\mathrm{e}^{\mathrm{i}\omega} - 2 + \mathrm{e}^{-\mathrm{i}\omega}).
$$

Using the formula $\cos\omega = 1 - 2\sin^2(\omega/2)$ we get

$$
\xi = \frac{1}{1 + 4r\sin^2\frac{\omega}{2}}.
$$

Since $r > 0$ we have $0 \leq \xi \leq 1$, therefore the scheme is stable.