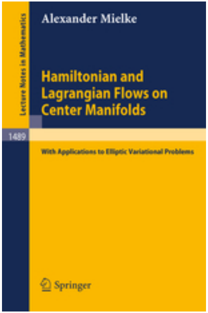
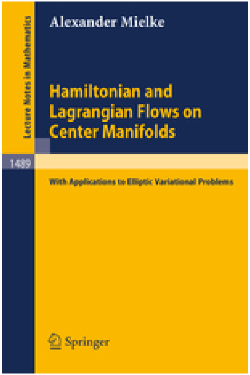
From the Benjamin-Feir instability to Whitham modulation theory and beyond

Thomas J. Bridges
joint work with Daniel Ratliff (Loughborough)



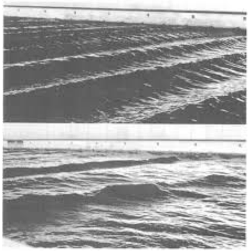
- 1. Symplectic center manifolds and BF instability
- 2. Implications of BF
  - The Benjamin-Feir index
  - Cascading generated by BF
- 3. New developments in Whitham modulation theory
  - Implications for BF and shallow water hydrodynamics

1. Symplectic center manifolds



- Circa 1990-1991
- $\leftarrow$ Symplecticity, dynamical systems, and water waves
- EU network: Heriot-Watt, Nice, Stuttgart, Utrecht
- Alexander von Humboldt
- Stuttgart $\rightarrow$ Hannover
- A proof of the BF instability

Benjamin-Feir instability



- circa 1967
- Periodic travelling wave becomes unstable to waves of different but close wavelength (sidebands)
- generally modelled by focusing NLS★

$$\mathrm{i}A_t + A_{xx} + 2|A|^2 A = 0\,.$$

- Stabilizes in shallow water: $\frac{L}{h_0} \approx 4.6$.

★ G. Biondini & D. Mantzavinos. *Universal nature of the nonlinear stage of modulational instability*, Phys. Rev. Lett. (2016).

Benjamin-Feir instability

Arch. Rational Mech Anal. 133 (1995) 145–198.

A Proof of the Benjamin-Feir Instability

THOMAS J. BRIDGES & ALEXANDER MIELKE

Communicated by P. HOLMES

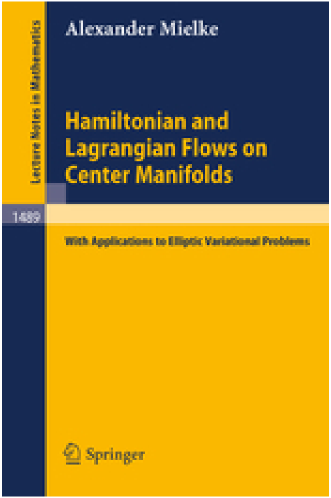
Abstract

The existence and linear stability problem for the Stokes periodic wavetrain on fluids of finite depth is formulated in terms of the spatial and temporal Hamiltonian structure of the water-wave problem. A proof, within the Hamiltonian framework, of instability of the Stokes periodic wavetrain is presented. A Hamiltonian center-manifold analysis reduces the linear stability problem to an ordinary differential eigenvalue problem on $\mathbb{R}^4$. A projection of the reduced stability problem onto the tangent space of the 2-manifold of periodic Stokes waves is used to prove the existence of a dispersion relation $\Lambda(\lambda, \sigma, I_1, I_2) = 0$ where $\lambda \in \mathbb{C}$ is the stability exponent for the Stokes wave with amplitude I_1 and mass flux I_2 and σ is the "sideband" or spatial exponent. A rigorous analysis of the dispersion relation proves the result, first discovered in the 1960's, that the Stokes gravity wavetrain of sufficiently small amplitude is unstable for $F \in (0, F_o)$ where $F_o \approx 0.8$ and F is the Froude number.

Table of contents

- Steady water waves; spatial Hamiltonian evolution; symplectic center manifold reduction
- second center manifold reduction for the linear stability problem
- Prove the existence of unstable eigenvalues
- Proof captures the change of stability at $kh_0 \approx 1.363$.
- To this day the only proof of BF for full water wave problem

Symplectic spatial dynamics



- Spatial dynamics still an active area
- Water waves; ferrofluids, vorticity, three-dimensionality
  - Mark Groves (Saarbrücken)
  - Kozlov, Lokharu (Linköping)
  - David Lloyd (Surrey)
  - Erik Wahlen (Lund) $\rightarrow$ ERC Fellowship

2. Implications of BF instability

Two new areas that have emerged which have strong dependence on the Benjamin-Feir instability

- The BFI: Benjamin Feir index, which is used to give probability maps of rogue wave frequency.
- Generation of energy cascades (Kartashova cascades) by repeated sideband instablilities starting with a BF instability.

The Benjamin-Feir index

A rough idea of the BFI can be obtained from the standard nonlinear Schrödinger model for BF instability

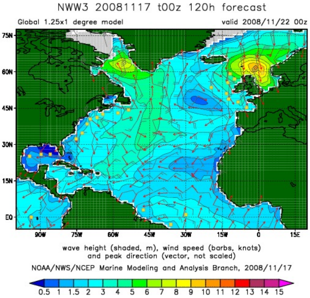
$$\mathrm{i}A_t + \tfrac{1}{2}\mathrm{i}c_p\nu A_x = \frac{1}{8}\frac{c_p}{k_0}\alpha A_{xx} + \frac{1}{2}c_p k_0^3\beta A|A|^2 = 0\,.$$

$$BFI = \frac{\varepsilon}{\Delta\omega/\omega_0}\nu\sqrt{\frac{\beta}{\alpha}}\,,$$

The spectral bandwidth $\Delta\omega$ is the most difficult to obtain and is estimated using ECMWF wave models.

★ M. Serio, M. Onorato, A.R. Osborne, & P.A.E.M. Janssen. *On the computation of the Benjamin-Feir index*, IL Nuovo Cimento (2005).

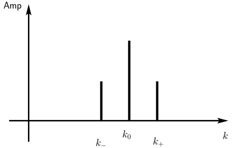
The Benjamin-Feir index



- $BFI = \frac{\varepsilon}{\Delta\omega/\omega_0}\nu\sqrt{\frac{\beta}{\alpha}}$
- $\varepsilon = k_0 a$ (wave steepness)
- $\Delta\omega = \frac{1}{2}\frac{\omega_0}{k_0}\Delta k$ (spectral bandwidth)
- high BFI $\Rightarrow$ non-Gaussian wave field
- ECMWF makes a map showing the distribution of BFI across the north Atlantic ocean, and sells it to shippers!

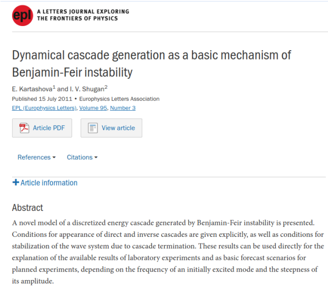
The Kartashova cascade

A schematic of the Ka-cascade is shown right. The sideband peaks are in the BF unstable region and they generate in turn two new sidebands. When they are also in the unstable region they generate further sidebands and so on.



This cascading phenomenon was first witnessed in the experiments of TULIN & WASEDA (1999). It is a *dynamic cascade* and a theory for it was proposed in KARTASHOVA ET AL. (2012).

The Kartashova cascade


A LETTERS JOURNAL EXPLORING THE FRONTIERS OF PHYSICS

Dynamical cascade generation as a basic mechanism of Benjamin-Feir instability

E. Kartashova[1] and I. V. Shugan[2]

Published 15 July 2011 • Europhysics Letters Association

EPL (Europhysics Letters), Volume 95, Number 3

Article PDF | View article

References | Citations

+ Article information

Abstract

A novel model of a discretized energy cascade generated by Benjamin-Feir instability is presented. Conditions for appearance of direct and inverse cascades are given explicitly, as well as conditions for stabilization of the wave system due to cascade termination. These results can be used directly for the explanation of the available results of laboratory experiments and as basic forecast scenarios for planned experiments, depending on the frequency of an initially excited mode and the steepness of its amplitude.

- Start with a BF instability
- Generate secondary sidebands
- Secondary to tertiary, etc
- Witnessed in experiments
- A theory has been proposed

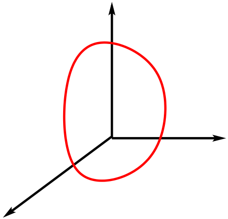
★ D. Dutykh & E. Tobisch. *Direct dynamical energy cascade in the modified KdV equation*, Physica D **297** 76–87 (2015).

Very interesting problem but precise mathematics lacking!

3. Whitham modulation theory

Consider an ODE
with a family of $T-$periodic solutions,

$$Z(t) = \widehat{Z}(\theta)\,, \quad \theta = \omega t\,.$$



WMT proceeds by
assuming a perturbed solution of the form

$$\widehat{Z}(\theta) \mapsto \widehat{Z}(\theta + \tfrac{1}{\varepsilon}\Theta) + \varepsilon^p W(\theta + \tfrac{1}{\varepsilon}\Theta, T)\,,$$

where T is a slow time variable and the modulation frequency is

$$\Omega := \frac{d\,\Theta}{dT}\,.$$

Perturb along the orbit as well as transverse to the orbit.
The perturbation $\theta \mapsto \theta + \frac{1}{\varepsilon}\Theta$ is called the geometric optics scaling.

ODEs: Kuramoto, Haken (1970s)
Phase diffusion equation: Manneville, Pomeau, Newell (1980s)
Phase Burger's equation: Doelman, Sandstede, Scheel, Schneider (2000s)

Whitham modulation theory

Given a conservative nonlinear set of PDEs in $1+1$

$$Z_t + \mathbf{L}Z + \mathbf{N}(Z) = 0\,,$$

with a two-parameter (ω, k) family of periodic travelling waves

$$Z(x,t) = \widehat{Z}(\theta)\,, \quad \theta = kx + \omega t\,.$$

WMT proceeds by assuming a perturbed solution of the form

$$\widehat{Z}(\theta) \mapsto \widehat{Z}(\theta + \tfrac{1}{\varepsilon}\Theta) + \varepsilon^p W(\theta + \tfrac{1}{\varepsilon}\Theta, X, T)\,, \qquad (\bigstar)$$

where X, T are slow space and time variables. The *modulation equations* are constructed via

- spatial averaging of conservation laws (Whitham 1965a)
- averaging the Lagrangian (Whitham 1965b)
- geometric optics ansatz $(\bigstar)$ (Luke 1967, Whitham 1970).

Whitham modulation theory

Governing equations are Euler-Lagrange equations.

The perturbation of $\widehat{Z}(\theta)$ is of the form

$$\widehat{Z}(\theta) \mapsto \widehat{Z}(\theta + \tfrac{1}{\varepsilon}\Theta) + \varepsilon^p W(\theta + \tfrac{1}{\varepsilon}\Theta, X, T)\,, \qquad (\bigstar)$$

where X, T are slow space and time variables. The WMEs are

$$q = \Theta_X \quad \text{and} \quad \Omega = \Theta_T \quad \Rightarrow \quad q_T = \Omega_X\,,$$

called conservation of waves, and

$$A(\Omega, q)_T + B(\Omega, q)_X = 0\,,$$

which is called conservation of wave action, with A and B deduced from the *averaged Lagrangian*.

Whitham modulation equations

The WMEs in 1+1 are based on the first order PDEs

$$A(u,v)_t + B(u,v)_x = 0$$

$$u_x = v_t$$

$$A_v = B_u \,.$$

For the scalar valued functions $u(x,t)$ and $v(x,t)$.

Whitham modulation equations

Differentiate A and B and write the WMEs in characteristic form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \frac{1}{A_u}\begin{bmatrix} 2A_v & B_v \\ -A_u & 0 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

There are two characteristics

$$c = \frac{-A_v \pm \sqrt{-\Delta_L}}{A_u} ,$$

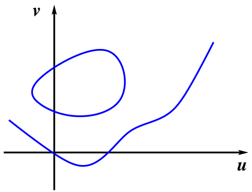
where

$$\Delta_L = \begin{bmatrix} A_u & A_v \\ B_u & B_v \end{bmatrix} .$$

$\Delta_L = 0$ signals a change of type from hyperbolic to elliptic.

The set $\Delta_L^{-1}(0)$

The set $\Delta_L^{-1}(0)$ is not necessarily connected and separates $(u, v)-$space into hyperbolic and elliptic regions.



$$\Delta_L \neq 0 \Rightarrow \begin{bmatrix} A_u & A_v \\ B_u & B_v \end{bmatrix} \begin{pmatrix} n_1 \\ n_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

$\mathbf{n} = (n_1, n_2)$ is a normal vector on the image of the mapping $(u, v) \mapsto (A(u, v), B(u, v))$, and defines an intrinsic second derivative.
cf. singularity theory for mappings of the plane to the plane (Whitney, Arnold)

Recapitulation: WMEs

The WMEs in 1+1 are based on the first order PDEs

$$A(u,v)_t + B(u,v)_x = 0$$

$$u_x = v_t$$

$$A_v = B_u \,.$$

For the scalar valued functions $u(x,t)$ and $v(x,t)$.

Key mapping is $(u,v) \mapsto (A(u,v), B(u,v))$ and its Jacobian

$$\Delta_L = \det \begin{bmatrix} A_u & A_v \\ B_u & B_v \end{bmatrix} \,.$$

Multiphase Whitham modulation equations

The multiphase WMEs are based on the vector-valued set of PDEs

$$\mathbf{A}(\mathbf{u},\mathbf{v})_t + \mathbf{B}(\mathbf{u},\mathbf{v})_x = \mathbf{0}$$

$$\mathbf{u}_x = \mathbf{v}_t$$

$$\mathrm{D}_{\mathbf{v}}\mathbf{A} = \left(\mathrm{D}_{\mathbf{u}}\mathbf{B}\right)^T \ .$$

For the vector-valued functions $\mathbf{u}(x,t) \in \mathbb{R}^N$ and $\mathbf{v}(x,t) \in \mathbb{R}^N$.

There are $2N$ characteristics which may be hyperbolic or elliptic.

Ablowitz & Benney (1970) $\rightsquigarrow$ integrable systems $\rightsquigarrow$ symmetric non-integrable systems (B & Ratliff, 2016+)

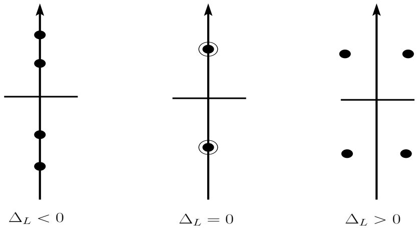
Scalar Whitham modulation theory

Going back to the scalar case

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \frac{1}{A_u}\begin{bmatrix} 2A_v & B_v \\ -A_u & 0 \end{bmatrix}\begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix} .$$

Linearize about $u = u_0$ and $v = v_0$ and let $(u, v) = (\hat{u}, \hat{v})e^{i\alpha x + \lambda t}$ then these eigenvalues satisfy

$$\lambda = \pm i c^{\pm} = \pm i \left(\frac{-A_v \pm \sqrt{-\Delta_L}}{A_u} \right) .$$



Collision of imaginary eigenvalues of opposite Krein signature?

WMT linearization

If you work at it you can find a symplectic structure! Thereby justifying the application of Krein signature to the above scenario, *but symplecticity is not the essence here*.

The WMEs, linearized about $\mathbf{u} = \mathbf{u}_0$ and $\mathbf{v} = \mathbf{v}_0$, are

$$\mathrm{D}_{\mathbf{u}}\mathbf{A}\,\mathbf{u}_t + \mathrm{D}_{\mathbf{v}}\mathbf{A}\,\mathbf{v}_t + \mathrm{D}_{\mathbf{u}}\mathbf{B}\,\mathbf{u}_x + \mathrm{D}_{\mathbf{v}}\mathbf{B}\,\mathbf{v}_x = \mathbf{0}\,.$$

$$\mathbf{u}_x = \mathbf{v}_t$$

Multiply the second equation by $\mathrm{D}_{\mathbf{u}}\mathbf{A}$ assuming it is invertible

$$\mathrm{D}_{\mathbf{u}}\mathbf{A}\,\mathbf{u}_x = \mathrm{D}_{\mathbf{u}}\mathbf{A}\,\mathbf{v}_t\,.$$

Krein signature $\longrightarrow$ sign characteristic

Introduce the normal mode ansatz

$$(\mathbf{u},\mathbf{v}) = (\hat{\mathbf{u}},\hat{\mathbf{v}})\mathrm{e}^{\mathrm{i}\alpha(x+ct)}\,.$$

Then rearranging the linearization,

$$\left[\underbrace{\begin{bmatrix} -\mathrm{D}_{\mathbf{u}}\mathbf{A} & 0 \\ 0 & \mathrm{D}_{\mathbf{v}}\mathbf{B} \end{bmatrix}}_{\textit{symmetric}} + c\underbrace{\begin{bmatrix} 0 & \mathrm{D}_{\mathbf{u}}\mathbf{A} \\ \mathrm{D}_{\mathbf{u}}\mathbf{A} & \mathrm{D}_{\mathbf{v}}\mathbf{A} + \mathrm{D}_{\mathbf{u}}\mathbf{B} \end{bmatrix}}_{\textit{symmetric}}\right]\begin{pmatrix} \widehat{\mathbf{u}} \\ \widehat{\mathbf{v}} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix}.$$

The linearized problem for characteristics in multiphase Whitham modulation theory is a Hermitian matrix pencil — relative to an indefinite metric:

$[\mathbf{F} + c\mathbf{G}]\mathbf{w} = \mathbf{0}$ with $\mathbf{F}^T = \mathbf{F}$ and $\mathbf{G}^T = \mathbf{G}$ and $\mathbf{G}$ indefinite.

Hermitian quadratic matrix polynomial

Let $\widehat{\mathbf{u}} = c\widehat{\mathbf{v}}$ then the HMP becomes

$$\mathbf{E}(c)\widehat{\mathbf{v}} := \left[\mathrm{D}_{\mathbf{u}}\mathbf{A}\, c^2 + (\mathrm{D}_{\mathbf{v}}\mathbf{A} + \mathrm{D}_{\mathbf{u}}\mathbf{B})c + \mathrm{D}_{\mathbf{v}}\mathbf{B}\right]\widehat{\mathbf{v}} = \mathbf{0}$$

A "Hermitian quadratic matrix polynomial". There is an extensive literature on this class of eigenvalue problems.

A key property that we will need is that a simple root, say c_0, has a "sign characteristic" defined by

$$\mathrm{S}(c_0) := \mathrm{sign}\left(\langle \xi, \mathbf{E}'(c_0)\xi \rangle \right),$$

where ξ is the eigenvector associated with c_0.

This sign is more general than Krein signature and predates it:

Hermitian matrix pencils (Weierstrass, 1868),

quadratic matrix polynomials (Gohberg, Lancaster, & Rodman, 1980).

The sign characteristic

- Simple eigenvalues of Hermitian linear/quadratic matrix polynomials have a sign characteristic★
- *Even zero eigenvalues* of Hermitian linear/quadratic matrix polynomials have a sign characteristic (in the symplectic case no such sign exists)
- Implication 1: The characteristics in linearized WMT have a sign characteristic.
- Implication 2: A necessary condition for two characteristics to transition from real to complex (hyperbolic to elliptic in WMT) is that they have opposite sign characteristic.

★ Lancaster & Rodman (2005), Tisseur et al. (2005+)

Breakdown of WMT when $\Delta_L = 0$

What happens in the nonlinear problem at coalescing characteristics?
— Dispersionless WMEs break down and we need to re-modulate.

We need to bring in a class of PDEs, generated by a Lagrangian, that has a class of wavetrain solutions, $\widehat{\Upsilon}(\theta, \omega, k)$, and re-think the derivation of the WMEs.

First, redo WMEs refining the geometric optics ansatz
(*dispersionless but not geometric optics*)

$$\Upsilon(x,t) = \widehat{\Upsilon}(\theta+\phi, \omega+\varepsilon\Omega, k+\varepsilon q) + \varepsilon^2 w(\theta+\phi, X, T, \varepsilon)\,.$$

with $X = \varepsilon x$ and $T = \varepsilon t$, gives at second order,

$$\mathscr{A}_\omega \Omega_T + \mathscr{A}_k q_T + \mathscr{B}_\omega \Omega_X + \mathscr{B}_k q_X = 0\,,$$

$$q_T - \Omega_X = 0\,.$$

Breakdown of WMT when $\Delta_L = 0$

The strategy at coalescence is to change the scaling and ansatz.

At the coalescence

$$c^{\pm} = -\frac{\mathscr{A}_k}{\mathscr{A}_\omega} := c_g \,.$$

The symbol c_g is used as it is a form of nonlinear group velocity. Try the modified ansatz

$$\Upsilon(x,t) = \widehat{\Upsilon}(\theta + \varepsilon\phi, \omega - \varepsilon^2 c_g q + \varepsilon^3 \Omega, k + \varepsilon^2 q) + \varepsilon^3 w(\theta, X, T, \varepsilon) \,.$$

with

$$X = \varepsilon(x - c_g t) \quad \text{and} \quad T = \varepsilon^2 t \,.$$

Breakdown of WMT when $\Delta_L = 0$

Substitute and solve order by order ... the orginal WMT ansatz

$$\widehat{\Upsilon}(\theta,\omega,k) \mapsto \widehat{\Upsilon}(\theta+\phi,\omega+\varepsilon\Omega,k+\varepsilon q) + \varepsilon^2 w(\theta+\phi,X,T,\varepsilon)\,.$$

generates

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \mathscr{A}_k q_X + \mathscr{B}_\omega \Omega_X + \mathscr{B}_k q_X = 0$$

The new $\Delta_L = 0$ ansatz

$$\widehat{\Upsilon}(\theta,\omega,k) \mapsto \widehat{\Upsilon}(\theta+\varepsilon\phi,\omega-c_g\varepsilon^2 q + \varepsilon^3\Omega,k+\varepsilon^2 q) + \varepsilon^3 w(\theta,X,T,\varepsilon)\,,$$

generates

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \kappa q q_X + \mathscr{K} q_{XXX} = 0$$

Breakdown of WMT when $\Delta_L = 0$

In the neighbourhood of $\Delta_L = 0$ the WMEs morph to

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \kappa q q_X + \mathscr{K} q_{XXX} = 0$$

which combine into

$$\mathscr{A}_\omega q_{TT} + \left(\tfrac{1}{2}\kappa q^2 + \mathscr{K} q_{XX}\right)_{XX} = 0$$

- Universal in the same sense that WMEs are universal
- Induced blow up in the modulation equation when $\mathscr{K}\mathscr{A}_\omega < 0$
- κ is the intrinsic second derivative of

$$(\omega, k) \mapsto \left(\mathscr{A}(\omega, k), \mathscr{B}(\omega, k)\right).$$

• TJB & D.J. Ratliff (2017) SIAM J. Appl. Math.

Extension to 2+1

The nonlinear theory for coalescing characteristics extends to $2+1$ where an additional term in the $Y-$direction arises

$$\mathscr{A}_\omega q_{TT} + \left(\tfrac{1}{2}\kappa q^2 + \mathscr{K} q_{XX}\right)_{XX} + \frac{\Delta_T}{\mathscr{A}_\omega} q_{YY} = 0\,,$$

where

$$\Delta_T = \det \begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_\ell \\ \mathscr{C}_\omega & \mathscr{C}_\ell \end{bmatrix}\,,$$

with wavenumbers (k, ℓ) and action flux components $(\mathscr{B}, \mathscr{C})$.

Another instability when $\Delta_T > 0$, in the $Y-$direction.

Can permute X, Y, T for other variants of this theory.

- TJB & D.J. Ratliff (2018) Phil. Trans.

Coalescing characteristics in the multiphase case

Characteristics, c, of the linearization of multiphase WMEs satisfy

$$0 = f(c) = \det\left[\mathbf{E}(c)\right] \,,$$

and coalescing characteristics, labelled c_g, satisfy

$$f(c_g) = f'(c_g) = 0 \quad \text{with} \quad \mathbf{E}(c_g)\zeta = 0\,.$$

Re-scale and modulate. In 1+1 case the result is

$$\alpha q_{TT} + \left(\tfrac{1}{2}\kappa q^2 + \mathscr{K} q_{XX}\right)_{XX} = 0\,.$$

Simlar to the case of single-phase wavetrain but the coefficients are more complicated; for example

$$\kappa = \frac{d^3}{ds^3}\mathscr{L}(\omega + s c_g \zeta, \mathbf{k} + s\zeta)\Big|_{s=0} \,,$$

This derivative formula is also an intrinsic derivative.

TJB & D.J. Ratliff (2018) preprint

Examples

- Modulation of multiphase wavetrains of coupled NLS equations

$$
\begin{aligned}
\mathrm{i}\frac{\partial \Psi_1}{\partial t} + \alpha_1 \frac{\partial^2 \Psi_1}{\partial x^2} + (\beta_{11}|\Psi_1|^2 + \beta_{12}|\Psi_2|^2)\Psi_1 &= 0 \\
\mathrm{i}\frac{\partial \Psi_2}{\partial t} + \alpha_2 \frac{\partial^2 \Psi_2}{\partial x^2} + (\beta_{21}|\Psi_1|^2 + \beta_{22}|\Psi_2|^2)\Psi_2 &= 0\,,
\end{aligned}
$$

- Modulation of Stokes waves and the critical transition in shallow water ($kh \approx 1.363$)
- Other examples in the literature

Classification by codimension

Dispersionless WMEs

↓

$$\mathscr{B}_k = 0$$

↓

$$(\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX} = 0$$

$\mathscr{A}_k = 0$:

$$\mathscr{A}_\omega q_{TT} + \left(\tfrac{1}{2}\mathscr{B}_{kk} q^2 + \mathscr{K} q_{XX}\right)_{XX} = 0$$

$\mathscr{B}_{kk} = 0$:

$$(\mathscr{A}_k + \mathscr{B}_\omega) q_T + \tfrac{1}{2}\mathscr{B}_{kkk} q^2 q_X + \mathscr{K} q_{XXX} = 0$$

$\mathscr{K} = 0$:

$$(\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_{kk} q q_X + \mathscr{K}_4 q_{XXXXX} = 0$$

... dual versions, 2+1 extensions, multiphase

... potential extensions to non-abelian, EP, etc

... modulation of relative equilibria

BF transition → 2-way Boussinesq equation

- Stokes waves coupled to mean flow is a two-phase wave train!
- WMT leads to multiphase WMEs (Whitham, 1967)
- the transition $kh_0 \approx 1.363$ is associated with coalescing characteristics
- new result: re-modulate → 2-way Boussinesq equation★

$$\alpha q_{TT} + \left(\tfrac{1}{2}\kappa q^2 + \mathscr{K} q_{XX}\right)_{XX} = 0\,.$$

- the standard 2-way Boussinesq equation in shallow water is NOT VALID
    - proved in G. Schneider & E. Wayne. Comm. Pure Appl. Math. (2000)
- the new 2-way Boussinesq equation is asymptotically valid and potentially rigorously valid

★ TJB & D.J. Ratliff. in preparation (2018).