Phase dynamics and water waves

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Whitham modulation theory

WMT starts with a Lagrangian formulation of field equations,

$$\mathscr{L}(U) = \iint \mathcal{L}(U_t, U_x, U)\,\mathrm{d}x\mathrm{d}t\,, \tag{1}$$

for some vector-valued field $U(x, t)$. Suppose $\exists$ a periodic TW

$$U(x, t) = \widehat{U}(\theta)\,, \quad \widehat{U}(\theta + 2\pi) = \widehat{U}(\theta)\,, \quad \theta = kx + \omega t\,, \tag{2}$$

of wavenumber k and frequency ω. Average the Lagrangian over the phase θ

$$\mathscr{L}(\omega, k) = \frac{1}{2\pi}\int_0^{2\pi} \mathcal{L}(\omega\widehat{U}_\theta, k\widehat{U}_\theta, \widehat{U})\,\mathrm{d}\theta\,. \tag{3}$$

WMT - II

Now suppose ω and k are slowly varying in some sense and can be expressed in terms of the gradient of the phase,

$$k = \theta_x\,, \quad \omega = \theta_t \quad \Rightarrow \quad k_t = \omega_x \quad \text{conservation of waves} \tag{4}$$

Substituting (4) into (3)

$$\mathscr{L} = \frac{1}{2\pi}\int_0^{2\pi} \widehat{\mathcal{L}}(\theta_t, \theta_x)\,\mathrm{d}\theta\,. \tag{5}$$

Noether's theorem gives conservation of wave action $\mathscr{A}_t + \mathscr{B}_x = 0$ with $\mathscr{A} = \mathscr{L}_\omega$ and $\mathscr{B} = \mathscr{L}_k$.

WMEs: $k_t = \omega_x$ and $\mathscr{A}(\omega, k)_t + \mathscr{B}(\omega, k)_x = 0\,.$

WMT – III

Introduce slow time and space scales: $T = \varepsilon t$ and $X = \varepsilon x$.

Take (ω, k) in a neighborhood of some fixed TW parameters,

$$\omega \mapsto \omega + \varepsilon\Omega(X, T, \varepsilon) \quad \text{and} \quad k \mapsto k + \varepsilon q(X, T, \varepsilon)\,.$$

Then to leading order, q and Ω satisfy

$$q_T = \Omega_X \quad \text{and} \quad \mathscr{A}_\omega \Omega_T + \mathscr{A}_k q_T + \mathscr{B}_\omega \Omega_X + \mathscr{B}_k q_X = 0\,. \qquad (6)$$

With (ω, k) fixed, this equation is linear and, with the assumption $\mathscr{A}_\omega \neq 0$, the linear WMEs can be written in the standard form

$$\begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{A}(\omega, k) \begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{7}$$

with

$$\mathbf{A}(\omega, k) = \frac{1}{\mathscr{A}_\omega} \begin{bmatrix} 0 & -\mathscr{A}_\omega \\ \mathscr{B}_k & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix}. \tag{8}$$

The WMEs (7) are *hyperbolic* if the eigenvalues of $\mathbf{A}(\omega, k)$ are real and *elliptic* if the eigenvalues of $\mathbf{A}(\omega, k)$ are complex.

Ellipticity is an indication that the basic state is unstable to long wave modulational instability.

$$\begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{A}(\omega, k) \begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

The roots of $\det[\lambda \mathbf{I} - \mathbf{A}(\omega, k)] = 0$ satisfy

$$\lambda = \frac{\mathscr{A}_\omega + \mathscr{B}_k}{2\mathscr{A}_\omega} \pm \frac{1}{\mathscr{A}_\omega}\sqrt{-\Delta}, \quad \Delta = \det \begin{bmatrix} \mathscr{A}_\omega & \mathscr{A}_k \\ \mathscr{B}_\omega & \mathscr{B}_k \end{bmatrix}.$$

This form for the eigenvalues highlights the Lighthill condition:

when $\Delta > 0$ the WMEs are elliptic and when $\Delta < 0$ the WMEs are hyperbolic.

WMT generalizations

- stability of water waves (WHITHAM 1967)
- inclusion of meanflow (“potential variables” in WMT)
- extension of WMT to 2+1 (HAYES 1970s)
- WMT for multiphase waves (ABLOWITZ & BENNEY 1970)
- shocks (phase jumps)
    - HOWE (1968) (WMT + Rankine–Hugoniot jump conditions)
    - OSTROVSKII (1968) “envelope shock waves”
    - YUE & MEI (1980) spatial defocussing NLS model

$$\mathrm{i}\Psi_Y + \Psi_{XX} - |\Psi|^2\Psi = 0\,.$$

    - PEREGRINE ET AL. (1980s) wave jumps and caustics

WMT generalizations

- Wavefront dislocations (discontinuities in wave crests)
  - BERRY ET AL. (1970s)
  - TANAKA (1995)
  - COSTE ET AL. (1999)
  - KARJANTO & VAN GROESEN (2007)
- Validity of WMT
  - DÜLL & SCHNEIDER (2009) (NLS $\to$ WMT)
  - BRONSKI, JOHNSON & ZUMBRUN (2010s) (gKdV $\to$ WMT)
  - NOBLE, RODRIGUES, BENZONI-GAVAGE (2010s)
- Dispersive shocks and undular bores
  - GRIMSHAW, EL, SMYTH (2000s)
    (SWEs with dispersion $\to$ WMT)

Phase modulation – dissipative systems

- Dissipative pattern formation
  - phase PDEs (Kuramoto-Tsuzuki, 1970s)
  - phase diffusion equation (Manneville-Pomeau, 1979)
  - Kuramoto-Sivashinsky equation (1970s)
  - Cross-Newell equation (1980s)
    - $\mathbf{k} = \nabla\Theta$ and embed spatial WMT in gradient flow
- Reaction-diffusion equations
  - Kopell-Howard (1970s)
  - Burgers' equation (travelling jumps)
  - Doelman-Scheel-Sandstede-Schneider (2000s)
  - Defects and dislocations (Haragus-Scheel 2000s)

Scaling and singularities

- Whitham modulation equations – generic case

scales $X = \varepsilon x$, $T = \varepsilon t$ (geometric optics scaling)

- Reaction diffusion equations – generic case

scales $X = \varepsilon x$, $T = \varepsilon^2 t \Rightarrow$ Burgers' equation

- non-generic case?

Example: Burgers' equation $\to$ Kuramoto-Sivashinsky

$$q_T + \omega''(k) q q_X = \nu q_{XX}\,, \; T = \varepsilon^2 t,\; X = \varepsilon x,\; q \sim \varepsilon\,,$$

with $\nu = \nu_1 \varepsilon^2$ changes to

$$q_T + \omega''(k) q q_X = \nu_1 q_{XX} + \nu_2 q_{XXXX}\,,$$

with $T = \varepsilon^4 t$, $X = \varepsilon x$, $q \sim \varepsilon^3$.

Modulation ansatz

General form of modulation ansatz (Doelman et al, 2009)

$$U(x,t) = \widehat{U}(\theta + \varepsilon^{a}\phi, k + \varepsilon^{b} q) + \varepsilon^{c} W(X,T,\varepsilon)$$

where

$$T = \varepsilon^{\alpha} t\,, \quad X = \varepsilon^{\beta} x\,,$$

with α, β and a, b, c dependent on singularity structure, and conservation of waves

$$q(X,T,\varepsilon) = \frac{\partial}{\partial X}\phi(X,T,\varepsilon)\,.$$

To be contrasted with classical method of multiple scales★

$$U(x,t) = \widehat{U}(\theta, k) + \varepsilon^{c}\widetilde{W}(X,T,\varepsilon)$$

★ WHITHAM, Multiple scales justification of WMT (1970).

Codimension one WMT

Now revisit WMT with singularities and new scalings

$$\begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{A}(\omega, k) \begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix} , \tag{9}$$

with

$$\mathbf{A}(\omega, k) = \frac{1}{\mathscr{A}_\omega} \begin{bmatrix} 0 & -\mathscr{A}_\omega \\ \mathscr{B}_k & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix} . \tag{10}$$

Codimension one singularity: simple zero characteristic speed

- $\mathscr{A}_\omega \neq 0$ but $\mathscr{B}_k = 0$
- $\mathscr{B}_k \neq 0$ but $\mathscr{A}_\omega = 0$

Breakdown of WMT when $\mathscr{A}_\omega \neq 0$ but $\mathscr{B}_k = 0$

- With $\mathscr{B}_k = 0$, expand $\mathscr{B}(\omega + \varepsilon\Omega, k + \varepsilon q)_X$ to the next order

$$\mathscr{A}_\omega \Omega_T + (\mathscr{A}_k + \mathscr{B}_\omega) q_T + \varepsilon \mathscr{B}_{kk} q q_X + \cdots = 0\,.$$

- Change scales $X = \varepsilon x$ and $T = \varepsilon^3 t$ and $q \sim \varepsilon^2$ (KdV scaling)
- Conservation of waves requires $\Omega \sim \varepsilon^4$
- With new scaling conservation of wave action morphs into

$$\mathscr{A}_\omega \underbrace{\Omega_T}_{\varepsilon^7} + \underbrace{(\mathscr{A}_\omega + \mathscr{B}_k) \underbrace{q_T}_{\varepsilon^5} + \mathscr{B}_{kk} \underbrace{q q_X}_{\varepsilon^5} + \mathscr{K} \underbrace{q_{XXX}}_{\varepsilon^5}}_{\text{leading order terms}} + \cdots = 0\,.$$

WMT with $\mathscr{B}_k = 0$

- Start with Lagrangian, symmetry, basic state
- Basic periodic TW: $\widehat{U}(\theta, k, \omega)$, with $\theta = kx + \omega t$
- Average Lagrangian: $\mathscr{L}(\omega, k)$
- Wave action $\mathscr{A} = \mathscr{L}_\omega$ and wave action flux $\mathscr{B} = \mathscr{L}_k$.
- modulate with scales $X = \varepsilon x$ and $T = \varepsilon^3 t$,

$$U(x,t) = \widehat{U}(\theta + \varepsilon\phi, k + \varepsilon^2 q, \omega + \varepsilon^4 \Omega) + \varepsilon^3 W(\theta, X, T, \varepsilon)\,.$$

  Substitute, expand and set ε^n, $n = 1, \ldots, 5$, terms to zero.
- Conservation of wave action morphs into the KdV equation

$$(\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX} = 0$$

Dual KdV with $\mathscr{A}_\omega = 0$

- Start with Lagrangian, symmetry, basic state
- Basic periodic TW: $\widehat{U}(\theta, k, \omega)$, with $\theta = kx + \omega t$
- Average Lagrangian: $\mathscr{L}(\omega, k)$
- Wave action $\mathscr{A} = \mathscr{L}_\omega$ and wave action flux $\mathscr{B} = \mathscr{L}_k$.
- modulate with $X = \varepsilon^3 x$ and $T = \varepsilon t$,

$$U(x,t) = \widehat{U}(\theta + \varepsilon\phi, k + \varepsilon^4 q, \omega + \varepsilon^2 \Omega) + \varepsilon^3 W(\theta, X, T, \varepsilon)\,.$$

- Conservation of wave action morphs into the KdV equation

$$(\mathscr{A}_k + \mathscr{B}_\omega)\Omega_X + \mathscr{A}_{\omega\omega}\Omega\Omega_T + \mathscr{K}\,\Omega_{TTT} = 0$$

NB. of interest near the change of superharmonic instability

Emergence of dispersion in WMT

- start with a Lagrangian with symmetry
  - assume existence of basic state: relative equilibrium
- substitute the ansatz into the Euler-Lagrange equation

$$U(x,t) = \widehat{U}(\theta + \varepsilon\phi, k + \varepsilon^2 q, \omega + \varepsilon^4 \Omega) + \varepsilon^3 W(\theta, X, T, \varepsilon) \, .$$

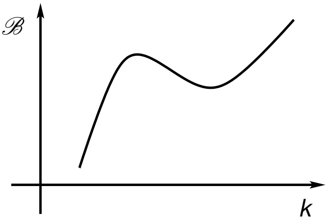
- solve order by order in ε
- third order solvable iff $\mathscr{B}_k = 0$
  - Jordan chain of length four generated
- Jordan chain terminates if $\mathscr{K} \neq 0$ (coefficient of dispersion)
- fifth order solvable iff q satisfies KdV

• TJB, Proc. Roy. Soc. Lond. A **469** (2014)

• TJB, Stud. Appl. Math. (in press, 2015)

RE and finding KdV emergent points

Along branches of periodic TWs, plot action flux $\mathscr{B}(\omega, k)$ versus k for fixed ω. Critical points signal potential points for KdV.



WMT is really about relative equilibria: RE are solutions with constant speed (or gradient) along a group direction.

$$U(x,t) = \mathrm{G}_{\theta(x,t)} \widehat{U}\,, \quad \nabla\theta = (\omega, k)\,.$$

Periodic TWs are a special case of relative equilibrium.

What about the KdV equation for water waves?

- Lagrangian (Luke's Lagrangian)
- basic state? relative equilibrium associated with mass conservation
  - $\eta = h_0$, $\phi = u_0 x + \phi_0$
  - $\mathscr{A}(u_0)$ is mass density, $\mathscr{B}(u_0)$ is mass flux,

Emergent KdV equation is

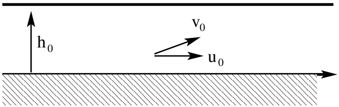
$$2\mathscr{A}_u q_T + \mathscr{B}_{uu} q q_X + \mathscr{K} q_{XXX} = 0$$

with

$$\mathscr{A}_u = -\frac{u_0}{g}\,, \quad \mathscr{B}_{uu} = -\frac{3u_0}{g}\,, \quad \mathscr{K} = \frac{1}{6} h_0^3\,.$$

The KdV equation found in textbooks.

- TJB, J. Fluid Mech. **761** (2014)



$$B = h_0 u_0 \quad \text{and} \quad C = h_0 v_0 \quad \text{(mass flux)}$$

$$R = gh_0 + \tfrac{1}{2}\left(u_0^2 + v_0^2\right) \quad \textbf{RE}: \; \phi = u_0 x + v_0 y + \phi_0\,.$$

Consider the generalized criticality conditions

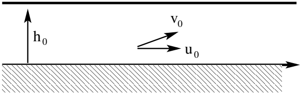
$$\mathscr{B}_u := \frac{\partial B}{\partial u_0}\bigg|_{R \text{ fixed}} = 0 \quad \text{and} \quad \mathscr{C}_u := \frac{\partial C}{\partial u_0}\bigg|_{R \text{ fixed}} = 0\,,$$

and the coefficients

$$\mathscr{B}_{uu} := \frac{\partial^2 B}{\partial u_0^2}\bigg|_{R \text{ fixed}} \quad \text{and} \quad \mathscr{C}_v := \frac{\partial C}{\partial v_0}\bigg|_{R \text{ fixed}}\,.$$

Modulation of uniform flows and KP

Modulate the uniform flow:



$$u_0 \mapsto u_0 + \varepsilon^2 q(X,Y,T,\varepsilon)\,, \quad X = \varepsilon x\,,\ Y = \varepsilon^2 x\,,\ T = \varepsilon^3 t\,.$$

q satisfies the KP equation with

$$\left(2\mathscr{A}_u \frac{\partial q}{\partial T} + \mathscr{B}_{uu} q \frac{\partial q}{\partial X} + \mathscr{K} \frac{\partial^3 q}{\partial X^3} \right)_X = \mathscr{C}_v q_{YY}\,.$$

It is a modulation of the mass CLAW for the full WW problem

$$A_t + B_x + C_y = 0\,,$$

where $\mathscr{A}$ is A evaluated on the uniform flow.

Water waves

Remarks

- contrast with the classical derivation
  - assume shallow water (gives SWEs)
  - assume "amplitude balances dispersion" (gives Boussinesq SWEs)
  - unidirectionalize (reduces Boussinesq SWEs to KdV$^{\pm}$)
- shallow water is neither necessary nor sufficient for the emergence of the KdV equation
- uniform flow gives coefficient of transverse dispersion

$$\left(2\mathscr{A}_u q_T + \mathscr{B}_{uu} q q_X + \mathscr{K} q_{XXX}\right)_X = \mathscr{C}_v q_{YY}$$

Emergence of KP – generalities

- Lagrangian $\int\int\int L\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t$, with one-parameter symmetry group
- symmetry $\Rightarrow$ CLAW $A_t + B_x + C_y = 0$
- basic state: $\widehat{Z}(\theta, k, \ell)$ with $\theta = kx + \ell y + \theta_0$, with singularity
- modulate: $Z = \widehat{Z}(\theta + \varepsilon\psi, k + \varepsilon^2 q, \ell + \varepsilon^3 L, \omega + \varepsilon^4 \Omega) + \varepsilon^3 W$
- Evaluate CLAW on basic state: $\mathscr{A}(k,\ell)$, $\mathscr{B}(k,\ell)$ and $\mathscr{C}(k,\ell)$
- With $\mathscr{B}_k = 0$ and $\mathscr{C}_k = 0$ KP emerges

$$(2\mathscr{A}_k q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX})_X = \mathscr{C}_\ell q_{YY}\,.$$

- $\mathscr{K}$: symplectic sign, Krein signature, sign of momentum flux, property of dispersion relation, etc

Modulation and 3+1 KP

Starting with a Lagrangian in $3+1$ with symmetry and a conservation law

$$A_t + B_x + C_y + D_z = 0\,.$$

Suppose the basic state is a relative equilibrium associated with this symmetry of the form

$$U(x,y,z,t) = \widehat{U}(\theta,k,\ell,m)\,, \quad \theta = kx + my + \ell z + \omega t\,.$$

If

$$\mathscr{B}_k = \mathscr{C}_k = \mathscr{D}_k = 0\,,$$

then, introduce the modulation

$$U = \widehat{U}(\theta + \varepsilon\phi, k + \varepsilon^2 q, \ell + \varepsilon^3 L, m + \varepsilon^3 M, \omega + \varepsilon^4 \Omega) + \varepsilon^3 W\,.$$

3+1 KP

With the usual assumptions and strategy, the governing equations are reduced to $3+1$ KP,

$$\Big((\mathscr{A}_k + \mathscr{B}_\omega) q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX} \Big)_X + \mathscr{C}_\ell q_{YY} + \mathscr{D}_m q_{ZZ} = 0 \,,$$

where the new directions are scaled like $Y = \varepsilon^2 y$ and $Z = \varepsilon^2 z$. The strategy extends in a natural way to $n+1$ KP for any natural number n.

Example: n+1 defocussing NLS can be reduced to n+1 KP

$$\mathrm{i}\Psi_t + \Psi_{xx} + \sum_{j=1}^{n-1} a_j \Psi_{x_j x_j} + \Psi - |\Psi|^2\Psi = 0 \,.$$

- D.J. Ratliff & TJB Preprint (2015)

WMT with codimension two singularity

$$\begin{pmatrix} q \\ \Omega \end{pmatrix}_T + \mathbf{A}(\omega,k)\begin{pmatrix} q \\ \Omega \end{pmatrix}_X = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad (11)$$

with

$$\mathbf{A}(\omega,k) = \frac{1}{\mathscr{A}_\omega}\begin{bmatrix} 0 & -\mathscr{A}_\omega \\ \mathscr{B}_k & \mathscr{A}_k + \mathscr{B}_\omega \end{bmatrix}. \qquad (12)$$

Umbilic point: nonsemisimple double zero characteristic speed

$$\mathbf{A}(\omega,k) = -\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

$\mathscr{A}_k = 0$ and $\mathscr{B}_k = 0$ but $\mathscr{A}_\omega \neq 0$

WMT with $\mathscr{A}_k = \mathscr{B}_k = 0$

- Start with Lagrangian, symmetry, conservation law
- basic periodic TW: $\widehat{U}(\theta, k, \omega)$, with $\theta = kx + \omega t$
- Average Lagrangian: $\mathscr{L}(\omega, k)$
- Wave action $\mathscr{A} = \mathscr{L}_\omega$ and wave action flux $\mathscr{B} = \mathscr{L}_k$.
- modulate with $X = \varepsilon x$ and $T = \varepsilon^2 t$,

$$U(x,t) = \widehat{U}(\theta + \varepsilon\phi, k + \varepsilon^2 q, \omega + \varepsilon^3 \Omega) + \varepsilon^3 W(\theta, X, T, \varepsilon)\,.$$

- To leading order q satisfies a two-way Boussinesq equation

$$\mathscr{A}_\omega q_{TT} + \left(\tfrac{1}{2}\mathscr{B}_{kk} q^2 + \mathscr{K} q_{XX}\right)_{XX} = 0\,.$$

• D.J. Ratliff & TJB Preprint (2015)

Apply to water waves.

Two-way Boussinesq model for water waves

For some scales X, T and coefficients a, b

$$\eta_{TT} - gh_0\eta_{XX} + \left(a\eta^2 + b\eta_{XX}\right)_{XX} = 0 \tag{13}$$

is derived for water waves in

- BOUSSINESQ (1872)
- URSELL (1953) (in the Lagrangian particle path setting)
- JOHNSON (1996) (also extends it to 2+1)

Application of the above umbilic modulation theory to uniform flows of the water wave problem with codimension two singularity shows that the equation (13) is asymptotically valid only in the limit $h_0 \to 0$!

Bousinesq SWEs $\rightarrow$ two-way Boussinesq

A simple derivation of the 2-way Boussinesq starts with a variant of the Boussinesq shallow water equations

$$\begin{aligned} h_t + hu_x + uh_x &= 0 \qquad (\star) \\ u_t + uu_x + gh_x &= \sigma h_{xxx} \end{aligned}$$

It is useful to add in the momentum conservation equation

$$(hu)_t + (hu^2 + \tfrac{1}{2}gh^2)_x = \sigma\big(hh_{xx} - \tfrac{1}{2}h_x^2\big)_x \,. \qquad (\star\star)$$

Differentiating ($\star$) with respect to t and combining with ($\star\star$),

$$h_{tt} - \left(hu^2 + \tfrac{1}{2}gh^2\right)_{xx} + \sigma(hh_{xxx})_x \,.$$

Two-way Boussinesq for water waves

$$h_{tt} - \left(hu^2 + \tfrac{1}{2} g h^2 \right)_{xx} + \sigma (h h_{xxx})_x \,.$$

Now let $h = h_0 + \eta$,

$$\eta_{tt} - g h_0 \eta_{xx} + \left(h_0 u^2 + \tfrac{1}{2} g \eta^2 \right)_{xx} + \sigma h_0 \eta_{xxxx} + \cdots \,.$$

Replace $u = \frac{1}{h_0} \int_x^{+\infty} \eta_t \, \mathrm{d}s$ and then introduce a transformation to eliminate the u^2 term, and neglect "higher order terms",

$$\underbrace{\eta_{tt}}_{\varepsilon^6} - \underbrace{g h_0 \eta_{xx}}_{\varepsilon^4} + \underbrace{\left(\tfrac{1}{2} g \eta^2 + \sigma h_0 \eta_{xx} \right)_{xx}}_{\varepsilon^6} = 0 \,.$$

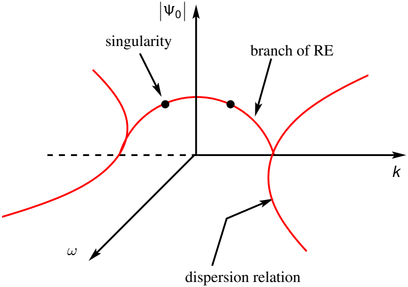
This is the version of the two-way Boussinesq equation found in the above papers and textbooks. It is not asymptotically valid!

WMT → Boussinesq

Nonlinear complex Klein-Gordon → two-way Boussinesq

$$\Psi_{tt} = \Psi_{xx} + \Psi - |\Psi|^2\Psi = 0 \,.$$

Dispersion relation for linear problem: $\omega^2 = k^2 - 1$.
Basic state: family of periodic TWs: $\Psi(x,t) = \Psi_0 \mathrm{e}^{\mathrm{i}(kx+\omega t)}$.



WMT $\to$ Boussinesq

Nonlinear complex Klein-Gordon $\to$ two-way Boussinesq

$$\Psi_{tt} = \Psi_{xx} + \Psi - |\Psi|^2\Psi = 0\,.$$

A point $\mathscr{A}_k = \mathscr{B}_k = 0$ exists on the branch of RE.
The two-way (good) Boussinesq equation emerges

$$\mathscr{A}_\omega q_{TT} + \left(\tfrac{1}{2}\mathscr{B}_{kk} q^2 + \mathscr{K}\, q_{XX}\right)_{XX} = 0\,, \quad \star$$

with

$$\omega = 0\,, \quad \mathscr{A}_\omega = -\frac{2}{3}\,, \quad \mathscr{B}_{kk} = -2\sqrt{3}\,, \quad \mathscr{K} = -\frac{1}{2}\,.$$

$\mathscr{A}_k = 0$ requires $\omega = 0$ and $\mathscr{B}_k = 0$ requires $k^2 = \frac{1}{3}$.

$\star$ extends in a natural way to $2+1$ Boussinesq

Water-waves and 2-way Boussinesq

Another way to look at the emergence of 2-way Boussinesq is to start with

$$U_t + F(U)_x = \mathbf{D} U_{xxx} \,, \quad U \in \mathbb{R}^n \,, \tag{14}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$, the flux vector, is a given smooth function.

Suppose $U = U_0$ is a constant vector and let

$$\mathbf{A} = DF(U_0) \,.$$

Suppose the constant $n \times n$ matrix $\mathbf{A}$ has a double nonsemisimple zero eigenvalue

$$\mathbf{A}\xi_1 = 0 \quad \text{and} \quad \mathbf{A}\xi_2 = \xi_1 \,.$$

Water waves and 2-way Boussinesq

Then, with the ansatz,

$$U(x,t) = U_0 + \varepsilon^2 q(X,T,\varepsilon)\xi_1 + \varepsilon^3 p(X,T,\varepsilon) + \varepsilon^4 W(X,T,\varepsilon)\,,$$

substitution into (14) reduces it, to leading order, to

$$q_T + p_X = 0 \quad \text{and} \quad p_T + \kappa q q_X = \nu q_{XXX}\,,$$

which combine to give the two-way Boussinesq equation.

Applications: two-layer and three-layer flow of differing densities in shallow water, shallow water with shear flow.

References: consistent with derivations of two-way Boussinesq using classical multiple scales in Hinkernell (1983), Helfrich & Pedlosky (1993), Mitsudera (1994), Grimshaw (1999)

Concluding remarks

- Singularities of multi-phase wavetrains
  - $n-$dimensional Abelian symmetry group and codimension 1 singlarity
    - – KdV coupled to $(n-1)-$dim generic wave action claw
- Pattern formation on the open ocean
  - time-dependent WMT in the plane
  - dislocations, phase jumps, etc.
- Theory based on Lagrangian with symmetry and RE
  - what about non-Abelian symmetry groups?