Rotation parameterizations conversions and their derivatives

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Abstract. This is a supplementary report to the TPAMI submission: HARD-PnP: PnP minimization using a Hybrid Approximate Representation [1]. It specifies the rules for conversions between different rotation parameterizations: ${}_i\underrightarrow{\mathrm{Rot}}_j$ and the Jacobians of these conversions: ${}_i\underrightarrow{\mathbf{J}}_j$. These are used during the joint optimization with robust overparameterization (see Equation (8) of the paper).
This supplementary report also presents an extension of this formalization to the second order terms, which could potentially be used for newton based non-linear solvers.

1 Parameterization definitions

Rotation vector:

$$\mathbf{RV} = (r_x, r_y, r_z)^\top , \tag{1}$$

where the direction of the vector represents the rotation axis and its amplitude is the rotation angle.

Euler axis and angle:

$$\mathbf{EA} = (a_x, a_y, a_z, a_m)^\top , \tag{2}$$

where the rotation axis $(a_x, a_y, a_z)^\top$ and the rotation angle a_m are separated.

Rotation quaternion:

$$\mathbf{Q} = (q_x, q_y, q_z, q_w)^\top , \tag{3}$$

encodes the 3D rotation matrix as a 4D complex number. Similar to the Euler axis, the first 3 terms relate to the orientation. However, the magnitude of the rotation is defined by the combination of all 4 elements (for a non-unit quaternion).

2 Parameterization conversions

Rotation vector to Axis-Angle ${}_{\mathbf{RV}}\underrightarrow{\mathrm{Rot}}_{\mathbf{EA}}$:

$$\mathbf{EA} = \left(\frac{r_x}{\sqrt{r_x^2 + r_y^2 + r_z^2}}, \frac{r_y}{\sqrt{r_x^2 + r_y^2 + r_z^2}}, \frac{r_z}{\sqrt{r_x^2 + r_y^2 + r_z^2}}, \sqrt{r_x^2 + r_y^2 + r_z^2} \right)^\top \tag{4}$$

Rotation vector to Quaternion $_{\mathbf{RV}}\underrightarrow{\mathrm{Rot}}_{\mathbf{Q}}$:

$$\mathbf{Q} = \begin{pmatrix} \frac{r_x \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} \\ \frac{r_y \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} \\ \frac{r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} \\ \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right) \end{pmatrix} \tag{5}$$

Axis-Angle to Rotation vector $_{\mathbf{EA}}\underrightarrow{\mathrm{Rot}}_{\mathbf{RV}}$:

$$\mathbf{RV} = (a_x a_m, a_y a_m, a_z a_m)^\top \tag{6}$$

Axis-Angle to Quaternion $_{\mathbf{EA}}\underrightarrow{\mathrm{Rot}}_{\mathbf{Q}}$:

$$\mathbf{Q} = \left(a_x \sin\left(\frac{a_m}{2}\right), a_y \sin\left(\frac{a_m}{2}\right), a_z \sin\left(\frac{a_m}{2}\right), \cos\left(\frac{a_m}{2}\right)\right)^\top \tag{7}$$

Quaternion to Rotation vector $_{\mathbf{Q}}\underrightarrow{\mathrm{Rot}}_{\mathbf{RV}}$:

$$\mathbf{RV} = \left(\frac{2q_x \cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}}, \frac{2q_y \cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}}, \frac{2q_z \cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}}\right)^\top \tag{8}$$

Quaternion to Axis-Angle $_{\mathbf{Q}}\underrightarrow{\mathrm{Rot}}_{\mathbf{EA}}$:

$$\mathbf{EA} = \left(\frac{q_x}{\sqrt{q_x^2+q_y^2+q_z^2}}, \frac{q_y}{\sqrt{q_x^2+q_y^2+q_z^2}}, \frac{q_z}{\sqrt{q_x^2+q_y^2+q_z^2}}, 2\cos^{-1}(q_w)\right)^\top \tag{9}$$

3 Jacobians of Conversions

3.1 Rotation vector to Axis-Angle

$$_{\mathbf{RV}}\underrightarrow{\mathbf{J}}_{\mathbf{EA}} = \begin{pmatrix} -\frac{r_x^2}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{1}{\sqrt{r_x^2+r_y^2+r_z^2}} & -\frac{r_x r_y}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} & -\frac{r_x r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \\ -\frac{r_x r_y}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} & -\frac{r_y^2}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{1}{\sqrt{r_x^2+r_y^2+r_z^2}} & -\frac{r_y r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \\ -\frac{r_x r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} & -\frac{r_y r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} & -\frac{r_z^2}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{1}{\sqrt{r_x^2+r_y^2+r_z^2}} \\ \frac{r_x}{\sqrt{r_x^2+r_y^2+r_z^2}} & \frac{r_y}{\sqrt{r_x^2+r_y^2+r_z^2}} & \frac{r_z}{\sqrt{r_x^2+r_y^2+r_z^2}} \end{pmatrix} \tag{10}$$

$$_{\mathbf{RV}}\underrightarrow{\mathbf{J}}_{\mathbf{EA}} = \begin{pmatrix} \frac{\partial a_x}{\partial r_x} & \frac{\partial a_x}{\partial r_y} & \frac{\partial a_x}{\partial r_z} \\ \frac{\partial a_y}{\partial r_x} & \frac{\partial a_y}{\partial r_y} & \frac{\partial a_y}{\partial r_z} \\ \frac{\partial a_z}{\partial r_x} & \frac{\partial a_z}{\partial r_y} & \frac{\partial a_z}{\partial r_z} \\ \frac{\partial a_m}{\partial r_x} & \frac{\partial a_m}{\partial r_y} & \frac{\partial a_m}{\partial r_z} \end{pmatrix} \tag{11a}$$

$$\frac{\partial a_x}{\partial r_x} = -\frac{r_x^2}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{1}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{11b}$$

$$\frac{\partial a_x}{\partial r_y} = -\frac{r_x r_y}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{11c}$$

$$\frac{\partial a_x}{\partial r_z} = -\frac{r_x r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{11d}$$

$$\frac{\partial a_y}{\partial r_x} = -\frac{r_x r_y}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{11e}$$

$$\frac{\partial a_y}{\partial r_y} = -\frac{r_y^2}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{1}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{11f}$$

$$\frac{\partial a_y}{\partial r_z} = -\frac{r_y r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{11g}$$

$$\frac{\partial a_z}{\partial r_x} = -\frac{r_x r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{11h}$$

$$\frac{\partial a_z}{\partial r_y} = -\frac{r_y r_z}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{11i}$$

$$\frac{\partial a_z}{\partial r_z} = -\frac{r_z^2}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{1}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{11j}$$

$$\frac{\partial a_m}{\partial r_x} = \frac{r_x}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{11k}$$

$$\frac{\partial a_m}{\partial r_y} = \frac{r_y}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{11l}$$

$$\frac{\partial a_m}{\partial r_z} = \frac{r_z}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{11m}$$

3.2 Rotation vector to Quaternion

$$_{\mathbf{RV}}\underset{\rightarrow}{\mathbf{J}}_{\mathbf{Q}} = \begin{pmatrix}
\frac{r_x^2 \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x^2 \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{\sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} & \frac{r_x r_y \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x r_y \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} & \frac{r_x r_z \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \\
\frac{r_x r_y \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x r_y \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} & \frac{r_y^2 \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_y^2 \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{\sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} & \frac{r_y r_z \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_y r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \\
\frac{r_x r_z \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} & \frac{r_y r_z \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_y r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} & \frac{r_z^2 \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_z^2 \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{\sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} \\
-\frac{r_x \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\sqrt{r_x^2+r_y^2+r_z^2}} & -\frac{r_y \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\sqrt{r_x^2+r_y^2+r_z^2}} & -\frac{r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\sqrt{r_x^2+r_y^2+r_z^2}}
\end{pmatrix} \tag{12}$$

$$
{}_{\mathbf{RV}}\underset{\rightarrow}{\mathbf{J}}_{\mathbf{Q}} = \begin{pmatrix} \frac{\partial q_x}{\partial r_x} & \frac{\partial q_x}{\partial r_y} & \frac{\partial q_x}{\partial r_z} \\ \frac{\partial q_y}{\partial r_x} & \frac{\partial q_y}{\partial r_y} & \frac{\partial q_y}{\partial r_z} \\ \frac{\partial q_z}{\partial r_x} & \frac{\partial q_z}{\partial r_y} & \frac{\partial q_z}{\partial r_z} \\ \frac{\partial q_w}{\partial r_x} & \frac{\partial q_w}{\partial r_y} & \frac{\partial q_w}{\partial r_z} \end{pmatrix} \tag{13a}
$$

$$
\frac{\partial q_x}{\partial r_x} = \frac{r_x^2 \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x^2 \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{\sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{13b}
$$

$$
\frac{\partial q_x}{\partial r_y} = \frac{r_x r_y \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x r_y \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{13c}
$$

$$
\frac{\partial q_x}{\partial r_z} = \frac{r_x r_z \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{13d}
$$

$$
\frac{\partial q_y}{\partial r_x} = \frac{r_x r_y \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x r_y \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{13e}
$$

$$
\frac{\partial q_y}{\partial r_y} = \frac{r_y^2 \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_y^2 \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{\sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{13f}
$$

$$
\frac{\partial q_y}{\partial r_z} = \frac{r_y r_z \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_y r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{13g}
$$

$$
\frac{\partial q_z}{\partial r_x} = \frac{r_x r_z \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_x r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{13h}
$$

$$
\frac{\partial q_z}{\partial r_y} = \frac{r_y r_z \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_y r_z \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} \tag{13i}
$$

$$
\frac{\partial q_z}{\partial r_z} = \frac{r_z^2 \cos\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\left(r_x^2+r_y^2+r_z^2\right)} - \frac{r_z^2 \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\left(r_x^2+r_y^2+r_z^2\right)^{3/2}} + \frac{\sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{\sqrt{r_x^2+r_y^2+r_z^2}} \tag{13j}
$$

$$
\frac{\partial q_w}{\partial r_x} = -\frac{r_x \sin\left(\frac{1}{2}\sqrt{r_x^2+r_y^2+r_z^2}\right)}{2\sqrt{r_x^2+r_y^2+r_z^2}} \tag{13k}
$$

$$\frac{\partial q_w}{\partial r_y} = -\frac{r_y \sin\left(\frac{1}{2}\sqrt{r_x^2 + r_y^2 + r_z^2}\right)}{2\sqrt{r_x^2 + r_y^2 + r_z^2}} \tag{13l}$$

$$\frac{\partial q_w}{\partial r_z} = -\frac{r_z \sin\left(\frac{1}{2}\sqrt{r_x^2 + r_y^2 + r_z^2}\right)}{2\sqrt{r_x^2 + r_y^2 + r_z^2}} \tag{13m}$$

3.3 Axis-Angle to Rotation vector

$$\mathbf{EA}\underset{\rightarrow}{\mathbf{J}}\mathbf{RV} = \begin{pmatrix} a_m & 0 & 0 & a_x \\ 0 & a_m & 0 & a_y \\ 0 & 0 & a_m & a_z \end{pmatrix} \tag{14}$$

3.4 Axis-Angle to Quaternion

$$\mathbf{EA}\underset{\rightarrow}{\mathbf{J}}\mathbf{Q} = \begin{pmatrix} \sin\left(\frac{a_m}{2}\right) & 0 & 0 & \frac{1}{2}a_x \cos\left(\frac{a_m}{2}\right) \\ 0 & \sin\left(\frac{a_m}{2}\right) & 0 & \frac{1}{2}a_y \cos\left(\frac{a_m}{2}\right) \\ 0 & 0 & \sin\left(\frac{a_m}{2}\right) & \frac{1}{2}a_z \cos\left(\frac{a_m}{2}\right) \\ 0 & 0 & 0 & -\frac{1}{2}\sin\left(\frac{a_m}{2}\right) \end{pmatrix} \tag{15}$$

3.5 Quaternion to Rotation vector

$$\mathbf{Q}\underset{\rightarrow}{\mathbf{J}}\mathbf{RV} = \begin{pmatrix} -\frac{2q_x^2\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{2\cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}} & -\frac{2q_xq_y\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{2q_xq_z\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{2q_x}{\sqrt{1-q_w^2}\sqrt{q_x^2+q_y^2+q_z^2}} \\ -\frac{2q_xq_y\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{2q_y^2\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{2\cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}} & -\frac{2q_yq_z\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{2q_y}{\sqrt{1-q_w^2}\sqrt{q_x^2+q_y^2+q_z^2}} \\ -\frac{2q_xq_z\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{2q_yq_z\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{2q_z^2\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{2\cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}} & -\frac{2q_z}{\sqrt{1-q_w^2}\sqrt{q_x^2+q_y^2+q_z^2}} \end{pmatrix} \tag{16}$$

$$\mathbf{Q}\underset{\rightarrow}{\mathbf{J}}\mathbf{RV} = \begin{pmatrix} \frac{\partial r_x}{\partial q_x} & \frac{\partial r_x}{\partial q_y} & \frac{\partial r_x}{\partial q_z} & \frac{\partial r_x}{\partial q_w} \\ \frac{\partial r_y}{\partial q_x} & \frac{\partial r_y}{\partial q_y} & \frac{\partial r_y}{\partial q_z} & \frac{\partial r_y}{\partial q_w} \\ \frac{\partial r_z}{\partial q_x} & \frac{\partial r_z}{\partial q_y} & \frac{\partial r_z}{\partial q_z} & \frac{\partial r_z}{\partial q_w} \end{pmatrix} \tag{17a}$$

$$\frac{\partial r_x}{\partial q_x} = -\frac{2q_x^2\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{2\cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}} \tag{17b}$$

$$\frac{\partial r_x}{\partial q_y} = -\frac{2q_xq_y\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} \tag{17c}$$

$$\frac{\partial r_x}{\partial q_z} = -\frac{2q_xq_z\cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} \tag{17d}$$

$$\frac{\partial r_x}{\partial q_w} = -\frac{2q_x}{\sqrt{1-q_w^2}\sqrt{q_x^2+q_y^2+q_z^2}} \tag{17e}$$

$$\frac{\partial r_y}{\partial q_x} = -\frac{2q_x q_y \cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} \tag{17f}$$

$$\frac{\partial r_y}{\partial q_y} = -\frac{2q_y^2 \cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{2\cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}} \tag{17g}$$

$$\frac{\partial r_y}{\partial q_z} = -\frac{2q_y q_z \cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} \tag{17h}$$

$$\frac{\partial r_y}{\partial q_w} = -\frac{2q_y}{\sqrt{1-q_w^2}\sqrt{q_x^2+q_y^2+q_z^2}} \tag{17i}$$

$$\frac{\partial r_z}{\partial q_x} = -\frac{2q_x q_z \cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} \tag{17j}$$

$$\frac{\partial r_z}{\partial q_y} = -\frac{2q_y q_z \cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} \tag{17k}$$

$$\frac{\partial r_z}{\partial q_z} = -\frac{2q_z^2 \cos^{-1}(q_w)}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{2\cos^{-1}(q_w)}{\sqrt{q_x^2+q_y^2+q_z^2}} \tag{17l}$$

$$\frac{\partial r_z}{\partial q_w} = -\frac{2q_z}{\sqrt{1-q_w^2}\sqrt{q_x^2+q_y^2+q_z^2}} \tag{17m}$$

3.6 Quaternion to Axis-Angle

$$\mathbf{_Q\underrightarrow{J}_{EA}} = \begin{pmatrix} -\frac{q_x^2}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{1}{\sqrt{q_x^2+q_y^2+q_z^2}} & -\frac{q_x q_y}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{q_x q_z}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & 0 \\ -\frac{q_x q_y}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{q_y^2}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{1}{\sqrt{q_x^2+q_y^2+q_z^2}} & -\frac{q_y q_z}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & 0 \\ -\frac{q_x q_z}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{q_y q_z}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} & -\frac{q_z^2}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{1}{\sqrt{q_x^2+q_y^2+q_z^2}} & 0 \\ 0 & 0 & 0 & -\frac{2}{\sqrt{1-q_w^2}} \end{pmatrix} \tag{18}$$

$$\mathbf{_Q\underrightarrow{J}_{EA}} = \begin{pmatrix} \frac{\partial a_x}{\partial q_x} & \frac{\partial a_x}{\partial q_y} & \frac{\partial a_x}{\partial q_z} & \frac{\partial a_x}{\partial q_w} \\ \frac{\partial a_y}{\partial q_x} & \frac{\partial a_y}{\partial q_y} & \frac{\partial a_y}{\partial q_z} & \frac{\partial a_y}{\partial q_w} \\ \frac{\partial a_z}{\partial q_x} & \frac{\partial a_z}{\partial q_y} & \frac{\partial a_z}{\partial q_z} & \frac{\partial a_z}{\partial q_w} \\ \frac{\partial a_m}{\partial q_x} & \frac{\partial a_m}{\partial q_y} & \frac{\partial a_m}{\partial q_z} & \frac{\partial a_m}{\partial q_w} \end{pmatrix} \tag{19a}$$

$$\frac{\partial a_x}{\partial q_x} = -\frac{q_x^2}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} + \frac{1}{\sqrt{q_x^2+q_y^2+q_z^2}} \tag{19b}$$

$$\frac{\partial a_x}{\partial q_y} = -\frac{q_x q_y}{\left(q_x^2+q_y^2+q_z^2\right)^{3/2}} \tag{19c}$$

$$\frac{\partial a_x}{\partial q_z} = -\frac{q_x q_z}{\left(q_x^2 + q_y^2 + q_z^2\right)^{3/2}} \tag{19d}$$

$$\frac{\partial a_x}{\partial q_w} = 0 \tag{19e}$$

$$\frac{\partial a_y}{\partial q_x} = -\frac{q_x q_y}{\left(q_x^2 + q_y^2 + q_z^2\right)^{3/2}} \tag{19f}$$

$$\frac{\partial a_y}{\partial q_y} = -\frac{q_y^2}{\left(q_x^2 + q_y^2 + q_z^2\right)^{3/2}} + \frac{1}{\sqrt{q_x^2 + q_y^2 + q_z^2}} \tag{19g}$$

$$\frac{\partial a_y}{\partial q_z} = -\frac{q_y q_z}{\left(q_x^2 + q_y^2 + q_z^2\right)^{3/2}} \tag{19h}$$

$$\frac{\partial a_y}{\partial q_w} = 0 \tag{19i}$$

$$\frac{\partial a_z}{\partial q_x} = -\frac{q_x q_z}{\left(q_x^2 + q_y^2 + q_z^2\right)^{3/2}} \tag{19j}$$

$$\frac{\partial a_z}{\partial q_y} = -\frac{q_y q_z}{\left(q_x^2 + q_y^2 + q_z^2\right)^{3/2}} \tag{19k}$$

$$\frac{\partial a_z}{\partial q_z} = -\frac{q_z^2}{\left(q_x^2 + q_y^2 + q_z^2\right)^{3/2}} + \frac{1}{\sqrt{q_x^2 + q_y^2 + q_z^2}} \tag{19l}$$

$$\frac{\partial a_z}{\partial q_w} = 0 \tag{19m}$$

$$\frac{\partial a_m}{\partial q_x} = 0 \tag{19n}$$

$$\frac{\partial a_m}{\partial q_y} = 0 \tag{19o}$$

$$\frac{\partial a_m}{\partial q_z} = 0 \tag{19p}$$

$$\frac{\partial a_m}{\partial q_w} = -\frac{2}{\sqrt{1 - q_w^2}} \tag{19q}$$

4 Hessians

In the main paper[1] we focus primarily on solutions using linear approximation. However, higher order terms in the Taylor expansion can also be kept. As an example we present here the formalization of the second order Hessian terms, including conversion to the reference representation.

In equation 9 of the main paper[1], the chain rule is used to convert the derivative of the composed conversion and rotation functions, into the product of their respective Jacobians. In non-matrix notation that is

$$\frac{\partial \mathrm{Rot}_j\left({}_1\underrightarrow{\mathrm{Rot}}_j\right)}{\partial \mathcal{R}} = \frac{\partial \mathrm{Rot}_j\left({}_1\underrightarrow{\mathrm{Rot}}_j\right)}{\partial {}_1\underrightarrow{\mathrm{Rot}}_j} \frac{\partial {}_1\underrightarrow{\mathrm{Rot}}_j}{\partial \mathcal{R}}. \tag{20}$$

It is easy to see how the multi-dimensional matrix form of this equation is simply the Jacobian product $\mathbf{J}_j {}_1\underrightarrow{\mathbf{J}}_j$. To obtain the second derivative, we can apply the product and chain rule to this result

$$\frac{\partial^2 \mathrm{Rot}_j\left({}_1\underrightarrow{\mathrm{Rot}}_j\right)}{\partial \mathcal{R}^2} = \frac{\partial^2 \mathrm{Rot}_j\left({}_1\underrightarrow{\mathrm{Rot}}_j\right)}{\partial {}_1\underrightarrow{\mathrm{Rot}}_j{}^2} \left(\frac{\partial {}_1\underrightarrow{\mathrm{Rot}}_j}{\partial \mathcal{R}}\right)^2 + \frac{\partial \mathrm{Rot}_j\left({}_1\underrightarrow{\mathrm{Rot}}_j\right)}{\partial {}_1\underrightarrow{\mathrm{Rot}}_j} \frac{\partial^2 {}_1\underrightarrow{\mathrm{Rot}}_j}{\partial \mathcal{R}^2}. \tag{21}$$

Extrapolating this to multi-dimensional matrix form is more complex than for the first order derivative. The first term can be fairly easily represented by combining the Hessian of the rotation function, with the Jacobian of the conversion function

$$\frac{\partial^2 \mathrm{Rot}_j\left({}_1\underrightarrow{\mathrm{Rot}}_j\right)}{\partial {}_1\underrightarrow{\mathrm{Rot}}_j{}^2} \left(\frac{\partial {}_1\underrightarrow{\mathrm{Rot}}_j}{\partial \mathcal{R}}\right)^2 \rightarrow {}_1\underrightarrow{\mathbf{J}}_j^\top \mathbf{H}_j {}_1\underrightarrow{\mathbf{J}}_j. \tag{22}$$

When multiplied on the right and left by $\begin{bmatrix} \Delta\mathcal{R} \\ \Delta\mathbf{t} \end{bmatrix}$ and it's transpose respectively as part of the Taylor series, it is easy to verify that this does indeed give us a measure of the total squared change in the output function for a given step in the parameter space. This makes intuitive sense, as we can imagine that the conversion Jacobian ${}_1\underrightarrow{\mathbf{J}}_j^\top$ takes an update in one parameter space, and gives us the (approximate) corresponding step in the reference parameterisation. The rotation Hessian $\mathbf{H}_j$ encodes the second order partial derivatives in the reference parameterization. Hence, when it is multiplied by every second order product of the steps in the reference parameterisation, the result is a single scalar encoding the "total derivative" estimate of the corresponding step in the output.

The second term in Equation 21 encodes the second derivative component due to the combined conversion and rotation functions $\frac{\partial \mathrm{Rot}_j \partial_1 \underrightarrow{\mathrm{Rot}}_j}{\partial \mathcal{R}^2}$. In multi-dimensional matrix form this is encoded using the conversion Hessian and the ith column of the rotation Jacobian

$$\frac{\partial \mathrm{Rot}_j\left({}_1\underrightarrow{\mathrm{Rot}}_j\right)}{\partial {}_1\underrightarrow{\mathrm{Rot}}_j} \frac{\partial^2 {}_1\underrightarrow{\mathrm{Rot}}_j}{\partial \mathcal{R}^2} \rightarrow \operatorname{diag}\left(\mathbf{J}_j\left[i\right]\right) {}_1\underrightarrow{\mathbf{H}}_j. \tag{23}$$

This results in a matrix containing every possible second order partial derivative of the form $\frac{\partial \mathrm{Rot}_j \partial \underrightarrow{{}_1\mathrm{Rot}_j}}{\partial \mathcal{R}^2}$. When this matrix is multiplied on the left and right by a step in the reference parameter space as in the Taylor expansion, the result is a scalar which corresponds to the "total derivative" step in the output space.

The actual form of the Hessians $\mathbf{H}_j$ and $\underrightarrow{{}_1\mathbf{H}_j}$ is too complex to easily present here. Instead we have provided Mathematica code to compute them as additional supplementary material.

References

1. Hadfield, S., Lebeda, K., Bowden, R.: HARD-PnP: PnP optimization using a hybrid approximate representation. In: Submitted to TPAMI. (2016) 1, 8