Car following model

* In heavy (but not light) traffic, drivers tend to follow each other.
* Acceleration / deceleration of one vehicle can be amplified in the system over time.

$$\underbrace{\frac{d^2 x_n(t+\tau)}{dt^2}}_{\text{acceleration}} = -\lambda \underbrace{\left(\frac{dx_n(t)}{dt} - \frac{dx_{n-1}(t)}{dt} \right)}_{\text{relative velocity, which is what you notice}}$$

x_{n-1} = position of leading vehicle
x_n = " " following car
τ = reaction time

Integrate once:

$$\frac{dx_n}{dt}(t+\tau) = -\lambda (x_n(t) - x_{n-1}(t)) + d_n$$

In a steady state all cars move at the same constant velocity. Assume also they are all the same distance apart. Then

$$\frac{dx_n(t+\tau)}{dt} = \frac{dx_n(t)}{dt}$$

so, if $d_n = d$,

$$\frac{dx_n(t)}{dt} = -\lambda (x_n(t) - x_{n-1}(t)) + d$$

Define traffic density ρ = no. of cars per unit length.
Then, over a distance L,

$$\text{no. of cars} = \rho L = \frac{L}{x_{n-1} - x_n}$$

$$\therefore \quad \rho = \frac{1}{x_{n-1} - x_n}$$

and, if velocity $u = \frac{dx_n}{dt}$, get $u = -\lambda\left(-\frac{1}{\rho}\right) + d$

$$u = \frac{\lambda}{\rho} + d$$

choose d so that $u = 0$ in maximum density situation (bumper to bumper traffic)

$\therefore$ want $u = 0$ when $\rho = \rho_{max}$

$$\therefore \quad u = \frac{\lambda}{\rho} - \frac{\lambda}{\rho_{max}} \quad \text{in heavy traffic}$$

(doesnt apply in light traffic since driver no longer following car in front).

Velocity $v_n = \frac{dx_n}{dt}$ given by

$$\frac{dv_n}{dt}(t+\tau) = -\lambda(v_n - v_{n-1})$$

Suppose $v_{n-1}(t) \equiv u$ but following vehicle has speed $v_n(t) = u + y(t)$. Then

$$\frac{dy}{dt}(t+\tau) = -\lambda(u + y(t) - u) = -\lambda y(t)$$

$$\text{or} \quad \frac{dy}{dt}(t) = -\lambda y(t-\tau)$$

Set $y(t) = e^{\sigma t}$ $\Rightarrow$ $\boxed{\sigma = -\lambda e^{-\sigma\tau}}$

- If $\lambda\tau \leq \frac{1}{e}$ then $\exists$ a <u>real</u> negative root σ (and all roots have $\mathrm{Re}\,\sigma < 0$)

  $\therefore$ expect $y(t) \to 0$ without oscillations.

- If $\frac{1}{e} < \lambda\tau < \frac{\pi}{2}$ then no real root but all roots still have $\mathrm{Re}\,\sigma < 0$

  $\therefore$ expect $y(t) \to 0$ with oscillations.

Now suppose leading vehicle has periodic velocity

$$v_0 = 1 + \tfrac{1}{2}\left(e^{i\omega t} + \underset{\substack{\uparrow \\ \text{complex conjugate}}}{c.c.}\right)$$

want to find n th driver's velocity $v_n(t)$.

Suppose

$$v_n(t) = 1 + \tfrac{1}{2}\left(f_n e^{i\omega t} + c.c.\right) \qquad \text{with } f_0 = 1.$$

Then, from

$$\frac{dv_n(t)}{dt} = -\lambda\left(v_n(t-\tau) - v_{n-1}(t-\tau)\right)$$

we get

$$\tfrac{1}{2} f_n i\omega e^{i\omega t} = -\lambda\left(\tfrac{1}{2} f_n e^{i\omega(t-\tau)} - \tfrac{1}{2} f_{n-1} e^{i\omega(t-\tau)}\right)$$

$$i\omega f_n = -\lambda\left(f_n e^{-i\omega\tau} - f_{n-1} e^{-i\omega\tau}\right)$$

$$f_n = \frac{\lambda f_{n-1} e^{-i\omega\tau}}{i\omega + \lambda e^{-i\omega\tau}}$$

$$\Rightarrow \quad f_n = \left[\frac{1}{1 + \frac{i\omega}{\lambda} e^{i\omega\tau}} \right] f_{n-1}$$

$$\therefore \quad f_n = \frac{1}{\left(1 + \frac{i\omega}{\lambda} e^{i\omega\tau}\right)^n} f_0 \qquad (\text{\& recall } f_0 = 1)$$

amplitude $|f_n|^2$ given by

$$|f_n|^2 = f_n \bar{f}_n = \frac{1}{\left(1 + \frac{i\omega}{\lambda} e^{i\omega\tau}\right)^n} \frac{1}{\left(1 - \frac{i\omega}{\lambda} e^{-i\omega\tau}\right)^n}$$

$$|f_n|^2 = \frac{1}{\left(1 + \frac{\omega^2}{\lambda^2} - \frac{2\omega}{\lambda} \sin\omega\tau\right)^n}$$

Need $|f_n|^2 \to 0$ as $n \to \infty$ so disturbance doesn't amplify.

$$\therefore \text{ need } \quad \frac{\omega^2}{\lambda^2} > \frac{2\omega}{\lambda} \sin\omega\tau$$

$$\text{or} \qquad \frac{\sin\omega\tau}{\omega} < \frac{1}{2\lambda}$$

But this holds for all ω if $\tau < \frac{1}{2\lambda}$

because then $\dfrac{\sin\omega\tau}{\omega} = \tau \dfrac{\sin\omega\tau}{\omega\tau} \leq \tau < \dfrac{1}{2\lambda}$

Recall τ = reaction time. If $\tau < \frac{1}{2\lambda}$ then disturbance goes away.

$$\tau = \begin{cases} 1 \text{ sec} & \text{without alcohol} \\ 1.5 \text{ secs} & \text{with 2 glasses of wine} \end{cases}$$

More alcohol | tiredness $\Rightarrow$ higher chance disturbance will amplify.