Calculus (Spring): Sheet 2

1. Show that $w = \ln(x^2 + y^2 + z^2)$ satisfies

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{2}{x^2 + y^2 + z^2}.$$

2. A horizontal plank of wood of width w and thickness h is supported at both ends, and x is the length between supports. The plank experiences a uniform load of p Newtons per metre. For example, think of a bookshelf loaded with many identical books. The plank will of course sag in the middle. The amount of sag S is calculated from the formula

$$S = C\,\frac{px^4}{wh^3}$$

where C is a constant depending on the type of wood. Show that, in the usual notation,

$$\delta S \approx C\left(\frac{x^4}{wh^3}\,\delta p + \frac{4px^3}{wh^3}\,\delta x - \frac{px^4}{w^2h^3}\,\delta w - \frac{3px^4}{wh^4}\,\delta h\right).$$

Suppose the plank is 90cm long, 27cm wide and 2cm thick and is subjected to a uniform load of 100 Nm^{-1}. Is the sag more sensitive to changes in thickness or to changes in width?

3. Show that a function $z = f(x, y)$ increases most rapidly in the direction of the vector ∇z, where $\nabla z = \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right)$, and decreases most rapidly in the direction of the vector $-\nabla z$. Use this result to find the path of steepest descent along the surface $z = x^2 + 3y^2$ from the point $(1, 1, 4)$ to the origin.
[Hint: start by noting that, at any point on the path, the vectors (dx, dy) and ∇z will be parallel].
Answer: the path's projection onto the (x, y) plane is $y = x^3$ from $(1, 1)$ to $(0, 0)$.

4. Find the stationary points of the function

$$f(x, y) = y^2 + xy + 3y + 2x + 3$$

and determine their nature. *Answer:* $(1, -2)$ *saddle.*

5. Show that the function

$$f(x, y) = xy^2 - 2y^3 - x^2 + 2xy$$

has stationary points at $(0, 0)$, $(\frac{3}{2}, 1)$ and $(4, 2)$ and determine their nature.

6. Find the stationary points of the function

$$f(x, y) = x^2 + y^2 + \frac{2}{xy}$$

and determine their nature. *Answer:* $(1, 1)$ *and* $(-1, -1)$, *both minimum.*

7. Find the dimensions of the most economical open-top rectangular crate $96\,\mathrm{m}^3$ in volume when the base costs 30 pence per square meter and the sides cost 10 pence per square meter. What is the actual minimum cost?

Please hand your work in at the lecture on Monday 15th March.
The lecture on Friday 12th will be used as a tutorial.