Calculus (Spring): Sheet 4

1. Show that

(i) $\displaystyle\iint_D \left(\frac{x-2y}{3x-y}\right) dA = \frac{8}{5}\ln 8$, where D is the parallelogram enclosed by the lines $x-2y = 0$, $x-2y=4$, $3x-y=1$ and $3x-y=8$;

(ii) $\displaystyle\iint_D \sqrt{4-x^2-y^2}\, dA = \frac{16\pi}{3}$, where D is the disk $x^2+y^2 \le 4$.

2. Show that

$$\int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1} x}{x}\, dx = \frac{\pi}{2}\ln\pi.$$

[Hint: Use the fact that $\tan^{-1}\pi x - \tan^{-1} x = \displaystyle\int_x^{\pi x} \frac{dy}{1+y^2}$ to write the left hand side as a *double* integral. Then sketch the region of integration and change the order of the integration.]

3. Show that

$$\int_0^2 \int_0^{z^2} \int_0^{\ln y} z\, e^x \, dx\, dy\, dz = \frac{4}{3}$$

4. Show that

$$\iiint_V \frac{1}{x^2+y^2}\, dx\, dy\, dz = 2\pi \ln 3$$

where V denotes the region between the cylinders $x^2+y^2=1$ and $x^2+y^2=9$ for $0 \le z \le 1$.

5. Show that the surface $z = \sqrt{x^2+y^2}$ is a cone, by showing that its equation in spherical polars is $\theta = \pi/4$. Sketch this cone together with the sphere $x^2+y^2+z^2 = 16$. Show that the volume of the region enclosed by the two surfaces is $\frac{64\pi}{3}(2-\sqrt{2})$.

6. If $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are constant vectors and $\mathbf{r} = (x,y,z)$ and V is the region described by $0 \le \mathbf{a}\cdot\mathbf{r} \le \alpha$, $0 \le \mathbf{b}\cdot\mathbf{r} \le \beta$, $0 \le \mathbf{c}\cdot\mathbf{r} \le \gamma$, show that

$$\iiint_V (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{r})\, dV = \frac{(\alpha\beta\gamma)^2}{8|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|}.$$

7. Let C be the boundary of any rectangular region in $\mathbf{R}^2$. Use Green's theorem to show that

$$\int_C (x^2y^3 - 3y)\, dx + x^3y^2\, dy$$

depends only on the area of the rectangle, and not where it is situated in $\mathbf{R}^2$.

8. Show, using Green's theorem, that

$$\int_C (3xy+y^2)\, dx + (2xy+5x^2)\, dy = 7\pi$$

where C is the curve $(x-1)^2+(y+2)^2 = 1$. [Hint: the interior of the circle with C as boundary can be described by $x = 1 + r\cos\theta$, $y = -2 + r\sin\theta$ for suitable ranges of r and θ].

Please hand your work in at the lecture on Friday 28th May.