Calculus (Spring) Sheet 1 solutions

1. (a) Characteristic equation is $m^2 - 2m + 17 = 0$ with roots $m = 1 \pm 4i$. So the solution is

$$y = e^x(A\cos 4x + B\sin 4x)$$

(b) Characteristic equation is $m^2 + 4m + 3 = 0$, i.e. $(m+3)(m+1) = 0$ so that $m = -3$ or -1. So the solution is

$$y = Ae^{-3x} + Be^{-x}$$

(c) Characteristic equation is $m^2 + 2m = 0$ with roots 0 and -2, giving the general solution to be

$$y = A + Be^{-2x}$$

Now $y(0) = 3$ so that $3 = A + B$. Also $y'(0) = -2$ so that $-2 = -2B$. Therefore $B = 1$ and $A = 2$ and the final solution is

$$y = 2 + e^{-2x}$$

2. (a) For the corresponding homogeneous problem the characteristic equation is $m^2 - 6m + 9 = 0$, i.e. $(m-3)^2 = 0$ so that there is one repeated root $m = 3$. The complementary solution y_c is therefore given by

$$y_c = (Ax + B)e^{3x}$$

For a particular solution y_p, try $y_p = ax + b$. Substituting into the original differential equation gives

$$-6a + 9(ax + b) = x$$

Comparing coefficients of x gives $a = \frac{1}{9}$, and comparing terms independent of x then gives $b = \frac{2}{27}$. So the particular solution is $y_p = \frac{1}{9}x + \frac{2}{27}$. The general solution of the differential equation is $y = y_c + y_p$ giving

$$y = (Ax + B)e^{3x} + \frac{1}{9}x + \frac{2}{27}$$

(b) Characteristic equation of the homogeneous problem is $m^2 - 4m + 8 = 0$ so that $m = 2 \pm 2i$ giving the complementary solution y_c to be

$$y_c = e^{2x}(A\cos 2x + B\sin 2x)$$

For the particular solution, try $y_p = Ae^{5x}$. Substituting this trial solution into the original differential equation gives

$$25Ae^{5x} - 20Ae^{5x} + 8Ae^{5x} = e^{5x}$$

so that $13A = 1$ giving $A = \frac{1}{13}$. Hence the particular solution is $y_p = \frac{1}{13}e^{5x}$ and the general solution y is given by $y = y_c + y_p$, i.e.

$$y = e^{2x}(A\cos 2x + B\sin 2x) + \frac{1}{13}e^{5x}$$

(c) Characteristic equation of the homogeneous problem is $m^2 + 2m + 2 = 0$ with roots $m = -1 \pm i$, so that the complementary solution is

$$y_c = e^{-x}(A\cos x + B\sin x)$$

For a particular solution, try $y_p = C\cos 3x + D\sin 3x$. Substituting this into the original differential equation gives

$$-9C\cos 3x - 9D\sin 3x + 2(-3C\sin 3x + 3D\cos 3x) + 2(C\cos 3x + D\sin 3x) = \sin 3x$$

Comparing $\cos 3x$ terms gives

$$-7C + 6D = 0$$

and comparing $\sin 3x$ terms gives

$$-7D - 6C = 1$$

The above simultaneous equations give $D = -\frac{7}{85}$ and $C = -\frac{6}{85}$, so the particular solution is

$$y_p = -\frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x$$

The general solution y is therefore

$$y = e^{-x}(A\cos x + B\sin x) - \frac{6}{85}\cos 3x - \frac{7}{85}\sin 3x$$

(d) Characteristic equation of the homogeneous problem is $m^2 + 6m + 8 = 0$, i.e. $(m+4)(m+2) = 0$ with roots -4 and -2 so that the complementary solution is

$$y_c = Ae^{-4x} + Be^{-2x}$$

The right hand side of the differential equation is $3e^{-2x}$. The complementary solution has an e^{-2x} term so the trial particular solution has to be $y_p = Cxe^{-2x}$ (rather than Ce^{-2x}). Differentiating this trial solution gives

$$\begin{aligned} y_p' &= -2Cxe^{-2x} + Ce^{-2x} \\ y_p'' &= 4Cxe^{-2x} - 2Ce^{-2x} - 2Ce^{-2x} = 4Cxe^{-2x} - 4Ce^{-2x} \end{aligned}$$

Substituting into the original differential equation gives

$$4Cxe^{-2x} - 4Ce^{-2x} + 6(-2Cxe^{-2x} + Ce^{-2x}) + 8Cxe^{-2x} = 3e^{-2x}$$

so that $2C = 3$ giving $C = \frac{3}{2}$. Hence the particular solution is $y_p = \frac{3}{2}xe^{-2x}$ and the general solution $y = y_c + y_p$ is given by

$$y = Ae^{-4x} + Be^{-2x} + \frac{3}{2}xe^{-2x}$$

Now $y(0) = 1$ so that

$$1 = A + B$$

Also $y'(0) = -3$ so that

$$-3 = -4A - 2B + \frac{3}{2}$$

Hence $A = \frac{5}{4}$ and $B = -\frac{1}{4}$. So the final solution is

$$y = \frac{5}{4}e^{-4x} - \frac{1}{4}e^{-2x} + \frac{3}{2}xe^{-2x}$$

3. Characteristic equation of homogeneous problem is

$$Lm^2 + Rm + \frac{1}{C} = 0$$

with roots

$$\begin{aligned} m &= \frac{1}{2L}\left(-R \pm \sqrt{R^2 - \frac{4L}{C}}\right) \qquad \text{but } L = \tfrac{CR^2}{2} \\ &= \frac{1}{CR^2}\left(-R \pm \sqrt{-R^2}\right) \\ &= \frac{1}{CR}(-1 \pm i) \end{aligned}$$

The complementary solution is therefore

$$q_c = e^{-t/CR}\left(A\cos\frac{t}{CR} + B\sin\frac{t}{CR}\right)$$

For the particular solution, the right hand side of the differential equation is constant so we try another constant, $q_p = D$ with D to be found. Substituting into the differential equation gives

$$\frac{1}{C}D = E$$

so that $D = CE$ and therefore $q_p = CE$. The general solution is $q = q_c + q_p$, i.e.

$$q = e^{-t/CR}\left(A\cos\frac{t}{CR} + B\sin\frac{t}{CR}\right) + CE$$