Fibonacci and Golden Ratio Formulae

Here are almost 200 formula involving the Fibonacci numbers and the golden ratio together with the Lucas numbers and the General Fibonacci series (the G series). This forms a major reference page for Ron Knott's Fibonacci Web site (http://www.mcs.surrey.ac.uk/Personal/R.Knott/Fibonacci/) where there are many more details and explanations with applications, puzzles and investigations aimed at secondary school students and teachers as well as interested mathematical enthusiasts.
Note that it is **easy to search for a named formula** on this page since it is an HTML page and the formulae are not images. In your browser main menu, under the **Edit** menu look for **Find...** and type Vajda-*N* or Dunlap-*N* for the relevant formula.

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Definitions and Notation

Beware of different golden ratio symbols used by different authors!
At this web site Phi is 1.618033... and phi is 0.618033.. but Vajda (see below) and Dunlap (see below) use a symbol for -0.618033.. .
Where a formula below (or a simple re-arrangement of it) occurs in either Vajda or Dunlap's book, the reference number they use is given. Dunlap's formulae are listed in his Appendix A3. Hoggatt's formula are from his "Fibonacci and Lucas Numbers" booklet. Full bibliographic details are at the end of this page.

As used here	**Vajda**	**Dunlap**	Description
floor(x)	[x]	trunc(x), not used for x<0	the nearest integer ≤ x. **When x>0**, this is "the integer part of x" or "truncate x" i.e. delete any fractional part after the decimal point. 3=floor(3)=floor(3.1)=floor(3.9), -4=floor(-4)=floor(-3.1)=floor(-3.9)
round(x)	$[x + \frac{1}{2}]$	trunc(x+1/2)	the nearest integer to x, equivalent to trunc(x+0.5) 3=round(3)=round(3.1), 4=round(3.9), -4=round(-4)=round(-3.9), -3=round(-3.1) 4=round(3.5), -3=round(-3.5)
ceil(x)	-	-	the nearest integer ≥ x. 3=ceil(3), 4=ceil(3.1)=ceil(3.9), -3=ceil(-3)=ceil(-3.1)=ceil(-3.9)

fract(x)	-	-	the fractional part of x, i.e. the part of abs(x) **after** the decimal point Knuth writes this a **x mod 1** defined as x–floor(x)
$\binom{n}{r}$	$\binom{n}{r}$	$\binom{n}{r}$	$= \frac{n!}{r!\,(n-r)!}$ $_nC_r$; n choose r; the element in row n column r of Pascal's Triangle; the coefficient of x^r in $(1+x)^n$; the number of ways of choosing r objects from a set of n different objects. n≥0 and r≥0.

F(i) is the Fibonacci series 0,1,1,2,3,5,... and L(i) is the Lucas series 2,1,3,4,7,11,....

Formula	**Refs**	Comments
$F(0) = 0, F(1) = 1,$ $F(n+2) = F(n+1) + F(n)$	-	Definition of the Fibonacci series
$F(-n) = (-1)^{n+1} F(n)$	Vajda-2, Dunlap-5	Extending the Fibonacci series 'backwards'
$L(0) = 2, L(1) = 1,$ $L(n+2) = L(n+1) + L(n)$	-	Definition of the Lucas series
$L(-n) = (-1)^n L(n)$	Vajda-4, Dunlap-6	Extending the Lucas series 'backwards'
$G(n+2) = G(n+1) + G(n)$	Vajda-3, Dunlap-4	Definition of the Generalised Fibonacci series, G(0) and G(1) needed
$\text{Phi} = 1.618... = \frac{\sqrt{5}+1}{2}$	Dunlap-63	Vajda and Dunlap use tau (τ) and Koshy uses alpha (α). Phi and –phi are the roots of $x^2 = x + 1$
$\text{phi} = 0.618... = \frac{\sqrt{5}-1}{2}$	Dunlap-65	Vajda uses –σ, and Dunlap uses –φ and Koshy uses –β Beware! Dunlap occasionally uses φ to represent our phi = 0.61803.., but more frequently he uses φ to represent -0.618033..

Linear Formulae

Linear relationships involve only sums or differences of Fibonacci numbers or Lucas numbers or their multiples.

Linear Sums of Fibonacci numbers

$F(n+3) + F(n) = 2\,F(n+2)$	-
$F(n+3) - F(n) = 2\,F(n+1)$	-
$F(n+4) + F(n) = 3\,F(n+2)$	-
$F(n+4) - F(n) = L(n+2)$	-
$F(n+6) + F(n) = 2\,L(n+3)$	-
$F(n+6) - F(n) = 4\,F(n+3)$	-
$F(n+1) + F(n-1) = L(n)$	Vajda-6, Hoggatt-18, Dunlap-14, Koshy-5.14
$F(n) + 2\,F(n-1) = L(n)$	(Dunlap-32)
$F(n+2) - F(n-2) = L(n)$	Vajda-7a, Dunlap-15, Koshy-5.15
$F(n+3) - 2\,F(n) = L(n)$	possible correction for Dunlap-31
$F(n+2) - F(n) + F(n-1) = L(n)$	possible correction for Dunlap-31
$F(n) + F(n+1) + F(n+2) + F(n+3) = L(n+3)$	C Hyson(*)

Linear Sums of Lucas numbers

$L(n-1) + L(n+1) = 5\,F(n)$	Vajda-5, Dunlap-13,

	Koshy-5.16
$L(n) + L(n + 3) = 2\,L(n + 2)$	-
$L(n) + L(n + 4) = 3\,L(n + 2)$	-
$2\,L(n) + L(n + 1) = 5\,F(n + 1)$	-
$L(n + 2) - L(n - 2) = 5\,F(n)$	-
$L(n + 3) - 2\,L(n) = 5\,F(n)$	-

Linear Sum of a Fibonacci and a Lucas number

$F(n) + L(n) = 2\,F(n + 1)$	Vajda-7b, Dunlap-16
$L(n) + 5\,F(n) = 2\,L(n + 1)$	-
$3\,F(n) + L(n) = 2\,F(n + 2)$	Vajda-26, Dunlap-28
$3\,L(n) + 5\,F(n) = 2\,L(n + 2)$	Vajda-27, Dunlap-29

Golden Ratio Formulae

Here Phi (see Definitions above) is Vajda's and Dunlap's τ)
and –phi (see Definitions above) is Vajda's σ, Dunlap's φ and Koshy's β.

Basic Phi Formulae

$\text{Phi}\ \text{phi} = 1$	Vajda page 51(3), Dunlap-65
$\text{Phi} + \text{phi} = \sqrt{5}$	-
$\text{Phi} / \text{phi} = \text{Phi} + 1$	-
$\text{phi} / \text{Phi} = 1 - \text{phi}$	-
$\text{Phi} - \text{phi} = 1$	-
$\text{Phi} = \text{phi} + 1 = \sqrt{5} - \text{phi}$	-
$\text{phi} = \text{Phi} - 1 = \sqrt{5} - \text{Phi}$	-
$\text{Phi}^2 = 1 + \text{Phi}$	Vajda page 51(4), Dunlap-64
$\text{phi}^2 = 1 - \text{phi}$	Vajda page 51(4), Dunlap-64
$\text{Phi}^{n+2} = \text{Phi}^{n+1} + \text{Phi}^n$	-
$(-\text{phi})^{n+2} = (-\text{phi})^{n+1} + (-\text{phi})^n$	-
$\text{phi}^n = \text{phi}^{n+1} + \text{phi}^{n+2}$	-
$(-\text{Phi})^n = (-\text{Phi})^{n+1} + (-\text{Phi})^{n+2}$	-

Golden Ratio with Fibonacci and Lucas

$F(n) = \dfrac{\text{Phi}^n - (-\text{phi})^n}{\sqrt{5}}$	**"Binet's" Formula** Vajda-58, Dunlap-69, Hoggatt-page 11, Binet(1843), De Moivre(1718), Lamé(1844)
$L(n) = \text{Phi}^n + (-\text{phi})^n$	Vajda-59, Dunlap-70
$F(n) = \text{round}\left(\dfrac{\text{Phi}^n}{\sqrt{5}}\right)$, if $n \ge 0$	Vajda-62, Dunlap-71 corrected
$L(n) = \text{round}(\text{Phi}^n)$, if $n \ge 2$	Vajda-63, Dunlap-72

$F(-n) = \text{round}\left(\frac{-(-\text{phi})^{-n}}{\sqrt{5}}\right)$, if $n \ge 0$	-
$L(-n) = \text{round}((-\text{phi})^{-n})$, $n \ge 3$	-
$F(-n) = (-1)^{n+1}\text{round}\left(\frac{\text{Phi}^n}{\sqrt{5}}\right)$, if $n \ge 0$	-
$F(n + 1) = \text{round}(\text{Phi } F(n))$, if $n \ge 2$	Vajda-64, Dunlap-73
$L(n + 1) = \text{round}(\text{Phi } L(n))$, if $n \ge 4$	Vajda-65, Dunlap-74
$\text{fract}(F(2n) \text{ phi}) = 1 - \text{phi}^{2n}$	Knuth vol 1, Ex 1.2.8 Qu 31
$\text{fract}(F(2n+1) \text{ Phi}) = \text{phi}^{2n-1}$	Knuth vol 1, Ex 1.2.8 Qu 31
$\text{Phi}^n = \frac{L(n) + F(n)\sqrt{5}}{2}$	Rabinowitz-25
$(-\text{phi})^n = \frac{L(n) - F(n)\sqrt{5}}{2}$	Rabinowitz-25
$\text{Phi}^n = \text{Phi } F(n) + F(n-1)$	Rabinowitz-28
$\text{Phi}^n = F(n+1) + F(n) \text{ phi}$	Rabinowitz-28
$\sqrt{5}\ \text{Phi}^n = \text{Phi } L(n) + L(n-1)$	-
$(-\text{phi})^n = -\text{phi } F(n) + F(n-1)$	Rabinowitz-28
$\sqrt{5}\ (-\text{phi})^n = \text{phi } L(n) - L(n-1)$	-
$(-\text{phi})^n = F(n+1) - \text{Phi } F(n)$	Vajda-103b, Dunlap-75
$L(n) + \sqrt{5}\ F(n) = 2\ \text{Phi}^n$	Vajda page 125
$L(n) - \sqrt{5}\ F(n) = 2\ (-\text{phi})^n$	Vajda page 125

Order 2 Formulae

Order 2 means these formula have a terms involving the *product of 2* Fibonacci or Lucas numbers at most.

Fibonacci numbers

$F(n)^2 + 2\ F(n-1)F(n) = F(2n)$	-
$F(n + 1)^2 + F(n)^2 = F(2n + 1)$	Vajda-11, Dunlap-7, Lucas(1876)
$F(n + 1)^2 - F(n - 1)^2 = F(2n)$	Lucas(1876)
$F(n + k + 1)^2 + F(n - k)^2 = F(2k + 1)F(2n + 1)$	a generalization of Vajda-11,Dunlap-7 Melham(1999)
$F(n + 2)\ F(n - 1) = F(n + 1)^2 - F(n)^2$	Vajda-12, Dunlap-8
$F(n + 1)\ F(n - 1) - F(n)^2 = (-1)^n$	Vajda-29, Dunlap-9, **Cassini's Formula**(1680), Simson(1753) special case of Catalan's Identity with r=1
$F(n)^2 - F(n + r)F(n - r) = (-1)^{n-r}F(r)^2$	**Catalan's Identity**(1879)
$F(n)F(m + 1) - F(m)F(n + 1) = (-1)^m F(n - m)$	**d'Ocagne's Identity**, special case of Vajda-9 with G=F
$F(n) = F(m)\ F(n + 1 - m) + F(m - 1)\ F(n - m)$	Dunlap-10
$F(n + m) = F(m)\ F(n + 1) + F(m - 1)\ F(n)$	alternative to Dunlap-10

$F(n)\ F(n+1) = F(n-1)\ F(n+2) + (-1)^{n-1}$	Vajda-20a special case: i:=1;k:=2;n:=n-1
$F(n+i)\ F(n+k) - F(n)\ F(n+i+k) = (-1)^n\ F(i)\ F(k)$	Vajda-20a=Vajda-18(corrected) with G:=H:=F
$F(a)(Fb) - F(c)F(d)$ $= (-1)^r(F(a-r)F(b-r) - F(c-r)F(d-r))$ a+b=c+d for any integers a,b,c,d,r	Johnson FQ 41 (2003) B-960, pg 182. Cassini, Catalan and D'Ocagne's Identities are all special cases of this formula

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F(nk) is a multiple of F(n)	-
gcd(F(m),F(n)) = F(gcd(m,n))	Lucas (1876)
F(m) mod F(n) = F(k)	Knuth Vol 1 Ex 1.2.8 Qu. 32

Lucas numbers

$L(2n) = L(n)^2 - 2\ (-1)^n$	-
$L(n+2)\ L(n-1) = L(n+1)^2 - L(n)^2$	-
$L(n+1)\ L(n-1) - L(n)^2 = -5\ (-1)^n$	-
$L(2n) + 2\ (-1)^n = L(n)^2$	Vajda-17c, Dunlap-12
$L(n+m) + (-1)^m\ L(n-m) = L(m)\ L(n)$	Vajda-17a, Dunlap-11

Fibonacci and Lucas Numbers

$F(2n) = F(n)\ L(n)$	Vajda-13, Hoggatt-17, Koshy-5.13
$L(n+1)^2 + L(n)^2 = 5\ F(2n+1)$	Vajda-25a
$L(n+1)^2 - L(n)^2 = 5\ F(2n)$	-
$L(n+1)^2 - 5\ F(n) = L(2n+1)^2$	-
$L(2n) - 2\ (-1)^n = 5\ F(n)^2$	Vajda-23, Dunlap-25
$F(n+1)\ L(n) = F(2n+1) + (-1)^n$	Vajda-30, Vajda-31, Dunlap-27, Dunlap-30
$L(n+1)\ F(n) = F(2n+1) - (-1)^n$	-
$F(2n+1) = F(n+1)\ L(n+1) - F(n)\ L(n)$	Vajda-14, Dunlap-18
$L(2n+1) = F(n+1)\ L(n+1) + F(n)\ L(n)$	-
$L(n)^2 - 2\ L(2n) = -5\ F(n)^2$	Vajda-22, Dunlap-24
$5\ F(n)^2 - L(n)^2 = 4\ (-1)^{n+1}$	Vajda-24, Dunlap-26
$F(n)^2 + L(n)^2 = = 4\ F(n+1)^2 - 2\ F(2n)$	FQ (2003)vol 41, B-936, M A Rose, page 87
$5\ (F(n)^2 + F(n+1)^2) = L(n)^2 + L(n+1)^2$	Vajda-25
$F(n)\ L(m) = F(n+m) + (-1)^m\ F(n-m)$	Vajda-15a, Dunlap-19
$L(n)\ F(m) = F(n+m) - (-1)^m\ F(n-m)$	Vajda-15b, Dunlap-20
$5\ F(m)\ F(n) = L(n+m) - (-1)^m\ L(n-m)$	Vajda-17b, Dunlap-23
$2\ F(n+m) = L(m)\ F(n) + L(n)\ F(m)$	Vajda-16a, Dunlap-21
$2\ L(n+m) = L(m)\ L(n) + 5\ F(n)\ F(m)$	-
$(-1)^m\ 2\ F(n-m) = L(m)\ F(n) - L(n)\ F(m)$	Vajda-16b, Dunlap-22
$L(n+i)\ F(n+k) - L(n)\ F(n+i+k) =$ $(-1)^{n+1}\ F(i)\ L(k)$	Vajda-19a

$F(n+i)\,L(n+k) - F(n)\,L(n+i+k) = (-1)^n F(i)\,L(k)$	Vajda-19b
$L(n+i)\,L(n+k) - L(n)\,L(n+i+k)$ $= (-1)^{n+1}\, 5\, F(i)\,F(k)$	Vajda-20b
$5F(a)F(b) - L(c)L(d) = (-1)^r(5F(a-r)F(b-r) - L(c-r)L(d-r))$ a+b=c+d for any integers a,b,c,d,r	Johnson

Higher Order Fibonacci and Lucas

$F(3n) = F(n+1)^3 + F(n)^3 - F(n-1)^3$	-
$[F(n-1)F(n+2)]^2 + [2F(n)F(n+1)]^2 = F(2n+1)^2$	A F Horadam *FQ* 20 (1982) pgs 121-122 special case of **Generalised Fibonacci Pythagorean Triples**
$[L(n-1)L(n+2)]^2 + [2L(n)L(n+1)]^2 = [5F(2n+1)]^2$	Wulczyn *FQ* 18 (1980) pg 188 special case of **Generalised Fibonacci Pythagorean Triples**
$F(n)^2\, F(m+1)\, F(m-1) - F(m)^2\, F(n+1)\, F(n-1)$ $= (-1)^{n-1}\, F(m+n)\, F(m-n)$	Vajda-32
$F(n+1)F(n+2)F(n+6) - F(n+3)^3 = (-1)^n F(n)$	FQ 41 (2003) pg 142, Melham
$F(n-2)F(n-1)F(n+1)F(n+2) + 1 = F(n)^4$	**Gelin-Cesàro Identity** (1880) FQ 41 (2003) pg 142.
$F(n)F(n+2)F(n+3)F(n+5) + 1 = [F(n+4)^2 - 2F(n+3)^2]^2$	
$F(i+j+k) =$ $F(i+1)F(j+1)F(k+1) + F(i)F(j)F(k) - F(i-1)F(j-1)F(k-1)$ for any integers i,j,k	Johnson's (6)
$\left(\frac{L(n) + \sqrt{5}\, F(n)}{2}\right)^k = \frac{L(kn) + \sqrt{5}\, F(kn)}{2}$	**De Moivre Analogue**
$\left(\frac{L(n) - \sqrt{5}\, F(n)}{2}\right)^k = \frac{L(kn) - \sqrt{5}\, F(kn)}{2}$	**De Moivre Analogue**
$(F(n)^2 + F(n+1)^2 + F(n+2)^2)^2 = 2 (F(n)^4 + F(n+1)^4 + F(n+2)^4)$	**Candido's Identity (1951)** FQ 42 (2004) R S Melham, pgs 155-160
$L(5n) = L(n)\,(L(2n) + 5F(n) + 3)(L(2n) - 5F(n) + 3)$, n odd	**Aurifeuille's Identity (1879)** FQ 42 (2004) R S Melham, pgs 155-160
$F(n)F(n+1)F(n+2)F(n+4)F(n+5)F(n+6) + L(n+3)^2 =$ $[F(n+3)(2F(n+2)F(n+4 - F(n+3)^2)]^2$	J Morgado **Note on some results of A F Horadam and A G Shannon concerning Catalan's Identity on Fibonaci Numbers** *Portugaliae Math.* 44 (1987) pgs 243-252

G Formulae

G(i) is the General Fibonacci series. It has the same recurrence relation as Fibonacci and Lucas, namely **G(n+2) = G(n+1) + G(n) for all integers n (i.e. n can be negative)**, but the "starting values" of G(0)=a and G(1)=b can be specified. It therefore includes both series them both as special cases. To make it clear which starting values for G(0)=a and G(1)=b are being used, we write G(a,b,i) for G(i). Hoggatt and others use the letter H for series G. For example:

- If G(0)=0 and G(1)=1 we have 0,1,1,2,3,5,8,13,.. the Fibonacci series, i.e. G(0,1,i) = F(i);
- G(0)=2 and G(1)=1 gives 2,1,3,4,7,11,18,.. the Lucas series, i.e. G(2,1,i) = L(i);

Basic G Formulae

$G(n+2) = G(n+1) + G(n)$	Vajda-3, Dunlap-4
$G(n) = G(0)\, F(n-1) + G(1)\, F(n)$	-
$G(-n) = (-1)^n\, (G(0)\, F(n+1) - G(1)\, F(n))$	-
$G(n+m) = F(m-1)\, G(n) + F(m)\, G(n+1)$	Vajda-8, Dunlap-33

$G(n-m) = (-1)^m (F(m+1)\,G(n) - F(m)\,G(n+1))$	Vajda-9, Dunlap-34
$L(m)\,G(n) = G(n+m) + (-1)^m G(n-m)$	Vajda-10a, Dunlap-35
$F(m)\,(G(n-1) + G(n+1)) = G(n+m) - (-1)^m G(n-m)$	Vajda-10b, Dunlap-36
$G(m)\,F(n) - G(n)\,F(m) = (-1)^{n+1} G(0)\,F(m-n)$	Vajda-21a
$G(m)\,F(n) - G(n)\,F(m) = (-1)^m G(0)\,F(n-m)$	Vajda-21b
$G(m+k)\,F(n+k) + (-1)^{k+1} G(m)\,F(n) = F(k)\,G(m+n+k)$	Howard(2003)

Order 2 G Formulae

These formulae include terms which are a product of two G numbers either from the same G series of from two different G series i.e. with different index 0 and 1 values. Where the series may be different they are denoted G and H e.g. special cases include G = F (i.e. Fibonacci) and H = L (i.e. Lucas), or they could also be the same series G=H.

$G(n+i)\,H(n+k) - G(n)\,H(n+i+k)$ $= (-1)^n (G(i)\,H(k) - G(0)\,H(i+k))$	Vajda-18 (corrected) a special case of Johnson's:
$G(p)H(q) - G(r)H(s)$ $= (-1)^n [\, G(p-n)H(q-n) - G(r-n)H(s-n) \,]$ if p+q = r+s and p,q,r,s,n are integers	Johnson (see ref below)
$G(n+1)\,G(n-1) - G(n)^2 = (-1)^n (G(1)^2 - G(0)\,G(2))$	Vajda-28
$\sqrt{5}\,G(n) = (G(1) + G(0)\,phi)\,Phi^n + (G(0)\,Phi - G(1))\,(-phi)^n$	Vajda-55/56, Dunlap-77
$G(i+j+k) = F(i+1)F(j+1)G(k+1) + F(i)F(j)G(k) - F(i-1)F(j-1)G(k-1)$ for any integers i,j,k	Johnson's (39a)
$4G(i)^2G(i+1)^2 + G(i-1)^2G(i+2)^2 = (\, G(i)^2 + G(i+1)^2 \,)^2$	Generalised Fibonacci Pythagorean Triples A F Horadam **Special Properties of the Sequence w_n(a,b;p,q)** *FQ* 5 (1967) pgs 424-434

Summations

This section has formulae that sum a variable number of terms.

Fibonacci and Lucas Summations

These formulae involve a sum of Fibonacci or Lucas numbers only.

$\sum_{i=0}^{n} F(i) = F(n+2) - 1$	Hoggatt-11, Lucas(1876)
$\sum_{i=0}^{n} L(i) = L(n+2) - 1$	Hoggatt-12
$\sum_{i=a}^{n} F(i) = F(n+2) - F(a+1)$	-
$\sum_{i=a}^{n} L(i) = L(n+2) - L(a+1)$	-
$\sum_{i=1}^{n} F(2i) = F(2n+1) - 1,\ n \geq 1$	Hoggatt-16, Lucas(1876)

$$\sum_{i=1}^{n} F(2i-1) = F(2n),\ n \geq 1$$ Hoggatt-15, Lucas(1876)

$$\sum_{i=1}^{n} L(2i-1) = L(2n)-2$$ -

$$\sum_{i=1}^{n} 2^{n-i} F(i-1) = 2^{n} - F(n+2)$$ Vajda-37a(adapted), Dunlap-42(adapted)

$$\sum_{i=0}^{n} (-1)^{i} L(n-2i) = 2\,F(n+1)$$ Vajda-97, Dunlap-54

Summations with fractions

$$\sum_{i=0}^{\infty} \frac{F(i}{2^{i}} = 2$$ Vajda-60, Dunlap-51

$$\sum_{i=0}^{\infty} \frac{L(i)}{2^{i}} = 6$$ -

$$\sum_{i=0}^{\infty} \frac{F(i)}{r^{i}} = \frac{r}{r^{2}-r-1}$$ -

$$\sum_{i=0}^{\infty} \frac{L(i)}{r^{i}} = 2 + \frac{r+2}{r^{2}-r-1}$$ -

$$\sum_{i=1}^{\infty} \frac{i\,F(i)}{2^{i}} = 10$$ Vajda-61, Dunlap-52

$$\sum_{i=1}^{\infty} \frac{i\,L(i)}{2^{i}} = 22$$ -

$$\sum_{i=1}^{\infty} \frac{1}{F(2^{i})} = 4 - Phi = 3 - phi$$ Vajda-77(corrected), Dunlap-53(corrected)

Order 2 summations

$$\sum_{i=1}^{2n} F(i)\,F(i-1) = F(2n)^{2}$$ Vajda-40, Dunlap-45

$$\sum_{i=1}^{2n} L(i)\,L(i-1) = L(2n)^{2} - 4$$ -

$\sum_{i=1}^{2n+1} F(i)\,F(i-1) = F(2n+1)^2 - 1$	Vajda-42, Dunlap-47
$\sum_{i=1}^{2n+1} L(i)\,L(i-1) = L(2n+1)^2 - 5$	-
$\sum_{i=0}^{n-1} F(2i+1)^2 = \frac{F(4n) + 2n}{5}$	Vajda-95
$\sum_{i=0}^{n-1} L(2i+1)^2 = F(4n) - 2n$	Vajda-96
$\sum_{i=1}^{n} F(i)^2 = F(n)\,F(n+1)$	Vajda-45, Dunlap-5, Hoggatt-13, Lucas(1876), Koshy-77
$\sum_{i=1}^{n} L(i)^2 = L(n)\,L(n+1) - 2$	Hoggatt-14
$\sum_{i=1}^{2n-1} L(i)^2 = 5\,F(2n)\,F(2n-1)$	-
$5\sum_{i=0}^{n} F(i)\,F(n-i) \begin{cases} = (n+1)\,L(n) - 2\,F(n+1) \\ = n\,L(n) - F(n) \end{cases}$	Vajda-98, Dunlap-55
$\sum_{i=0}^{n} L(i)\,L(n-i) \begin{cases} = (n+1)\,L(n) + 2\,F(n+1) \\ = (n+2)\,L(n) + F(n) \end{cases}$	Vajda-99, Dunlap-56
$\sum_{i=0}^{n} F(i)\,L(n-i) = (n+1)\,F(n)$	Vajda-100, Dunlap-57
$\sum_{i=1}^{n} L(2i)^2 = F(4n+2) + 2n - 1$	Vajda page 70

G Summations

$\sum_{i=1}^{n} G(i) = G(n+2) - G(2)$	Vajda-33, Dunlap-38
$\sum_{i=a}^{n} G(i) = G(n+2) - G(a+1)$	-
$\sum_{i=1}^{n} G(2i-1) = G(2n) - G(0)$	Vajda-34, Dunlap-37
$\sum_{i=1}^{n} G(2i) = G(2n+1) - G(1)$	Vajda-35, Dunlap-39

$\sum_{i=1}^{n} G(2i) - \sum_{i=1}^{n} G(2i-1) = G(2n-1) + G(0) - G(1)$	Vajda-36, Dunlap-40
$\sum_{i=1}^{n} 2^{n-i} G(i-1) = 2^{n-1}(G(0) + G(3)) - G(n+2)$	Vajda-37(variant), Dunlap-41(variant)
$\sum_{i=1}^{4n+2} G(i) = L(2n+1)\, G(2n+3)$	Vajda-38, Dunlap-43
$\sum_{i=1}^{2n} G(i)\, G(i-1) = G(2n)^2 - G(0)^2$	Vajda-39, Dunlap-44
$\sum_{i=1}^{2n+1} G(i)\, G(i-1) = G(2n+1)^2 - G(0)^2 - G(1)^2 + G(0)G(2)$	Vajda-41, Dunlap-46
$\sum_{i=1}^{n} G(i+2)\, G(i-1) = G(n+1)^2 - G(1)^2$	Vajda-43, Dunlap-48
$\sum_{i=1}^{n} G(i)^2 = G(n)\, G(n+1) - G(0)\, G(1)$	Vajda-44, Dunlap-49
$\sum_{i=0}^{\infty} \frac{G(a, b, i)}{r^i} = a + \frac{a + b r}{r^2 - r - 1}$	Stan Rabinowitz, "Second-Order Linear Recurrences" card, *Generating Function* special case (x=1/r, P=1, Q=-1)
$\sum_{i=0}^{\infty} \frac{i\, G(a, b, i)}{r^i} = \frac{r (b r^2 - 2 a r + b - a)}{(r^2 - r - 1)^2}$	-

Summations with Binomial Coefficients

$\sum_{i=1}^{n} \binom{n-i}{i-1} = F(n)$	-
$\sum_{i=0}^{\infty} \binom{n-i-1}{i} = F(n)$	Vajda-54(corrected), Dunlap-84(corrected)
$\sum_{i=0}^{n} \binom{n+1}{i+1} F(i) = F(2n+1) - 1$	Vajda-50, Dunlap-82
$\sum_{i=0}^{2n} \binom{2n}{i} F(2i) = 5^n\, F(2n)$	Vajda-69, Dunlap-85
$\sum_{i=0}^{2n} \binom{2n}{i} L(2i) = 5^n\, L(2n)$	Vajda-71, Dunlap-87

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(2i) = 5^n L(2n+1)$$ Vajda-70, Dunlap-86

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(2i) = 5^{n+1} F(2n+1)$$ Vajda-72, Dunlap-88

$$\sum_{i=0}^{2n} \binom{2n}{i} F(i)^2 = 5^{n-1} L(2n)$$ Vajda-73, Dunlap-89

$$\sum_{i=0}^{2n} \binom{2n}{i} L(i)^2 = 5^n L(2n)$$ Vajda-75, Dunlap-91

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(i)^2 = 5^n F(2n+1)$$ Vajda-74, Dunlap-90

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(i)^2 = 5^{n+1} F(2n+1)$$ Vajda-76, Dunlap-92

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i+1} = 2^{n-1} F(n)$$ Vajda-91

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i} = 2^{n-1} L(n)$$ Vajda-92

$$\sum_{i=0}^{k} \binom{k}{i} F(n)^i F(n-1)^{k-i} F(i) = F(kn)$$ Rabinowitz-17

$$\sum_{i=0}^{k} \binom{k}{i} F(n)^i F(n-1)^{k-i} L(i) = L(kn)$$ Rabinowitz-17

Summations with Binomials and G Series

$$\sum_{i=0}^{n} \binom{n}{i} G(i) = G(2n)$$ Vajda-47, Dunlap-80

$$\sum_{i=0}^{n} \binom{n}{i} G(p-i) = G(p+n)$$ Vajda-46, Dunlap-79

$$\sum_{i=0}^{n} \binom{n}{i} G(p+i) = G(p+2n)$$ Vajda-49, Dunlap-81

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} G(n+p-i) = G(p-n)$$ Vajda-51, Dunlap-83

Other Formulae

$$F(n) = \prod_{k=0}^{floor((n-1)/2)} \left(3 + 2\cos\frac{2k\pi}{n}\right) \ -$$

Hyperbolic Functions

Here we use g for ln(Phi), the natural log of Phi. cosh(g)=√5 / 2. There are several derivations of formulae above using hyperbolic functions in chapter XI of Vajda.

$$F(2n) = \frac{2}{\sqrt{5}}\sinh(2ng)$$ from Binet's formula

$$= \frac{\sinh(2ng)}{\cosh(g)}$$

$$F(2n+1) = \frac{2}{\sqrt{5}}\cosh((2n+1)g)$$ from Binet's formula

$$= \frac{\cosh((2n+1)g)}{\cosh(g)}$$

$L(2n) = 2\cosh(ng)$ from Binet's formula

$L(2n+1) = 2\sinh(ng)$ from Binet's formula

Complex Numbers

$$F(n) = \frac{2\, i^{1-n}}{\sqrt{5}}\sin(-i\, n \ln(i\, Phi))$$ from Rabinowitz-7 corrected

$$F(n) = \frac{2\, i^{-n}}{\sqrt{5}}\sinh(n \ln(i\, Phi))$$ from Rabinowitz-7 corrected

$L(n) = 2\, i^{-n} \cos(-i\, n \ln(i\, Phi))$ from Rabinowitz-7 corrected

$L(n) = 2\, i^{-n} \cosh(n \ln(i\, Phi))$ from Rabinowitz-7 corrected

References

(*) above indicates a private communication.

: a book;

: an article (chapter, paper) in a book (journal);

: a web resource.

FQ : The Fibonacci Quarterly

Arranged in alphabetical order of author:

R A Dunlap, The Golden Ratio and Fibonacci Numbers World Scientific Press, 1997, 162 pages.
An introductory book strong on the geometry and natural aspects of the golden section but it does not include much on the mathematical detail. Beware - some of the formula in the Appendix are wrall of the original's errors! The formulae on the page you are now reading are corrected versions and have been verified.

F T Howard (2003) "The Sum of the Squares of Two Generalized Fibonacci Numbers" *FQ* vol 41 pages 80-84.

V E Hoggatt Jr "Fibonacci and Lucas Numbers" published by The Fibonacci Association, 1969

(Houghton Mifflin).
A very good introduction to the Fibonacci and Lucas Numbers written by a founder of the Fibonacci Quarterly.

R Johnson (Durham university) has an excellent web page
on the power of matrix methods to establish many Fibonacci formula with ease (but it does rely on at least undergraduate level matrix mathematics). See the **Matrix methods for Fibonacci and Related Sequences** link to a Postscript and PDF version on his Fibonacci Resources web page. The latest version (Nov 12, 2004) contains an appendix showing how formulae developed in Johnson's paper can prove almost all the identities here in my table above.

D E Knuth The Art of Computer Programming: Vol 1 Fundamental Algorithms hardback, Addison-Wesley third edition (1997).
The paperback is now out of print and hard to find. This is the first of three volumes and an absolute must for all computer scientist/mathematicians.

T Koshy Fibonacci and Lucas Numbers with Applications, Wiley-Interscience, 2001, 648 pages.
This is a new book packed full of an amazing number of Fibonacci and related equations, culled from the pages of the *Fibonacci Quarterly*. Although initially impressive in its size and breadth, be aware that there are far too many typos, errors and missing or irrelevant conditions in many of its formulae as well as some glaring omissions and misattributions particularly with respect to the original references for a number of the formulae. Although Fibonacci representations of integers are included Zeckendorf himself is never mentioned! There are lots of exercises with answers to the odd-numbered questions.

E Lucas, "Théorie des Fonctions Numériques Simplement Périodiques" in *American Journal of Mathematics* vol 1 (1878) pages 184-240 and 289-321.
Reprinted as The Theory of Simply Periodic Functions, the Fibonacci Association, 1969.

R S Melham (1999) "Families of Identities Involving Sums of Powers of the Fibonacci and Lucas Numbers" *FQ* vol 37, pages 315-319.

S Rabinowitz "Algorithmic Manipulation of Fibonacci Identities" in Applications of Fibonacci Numbers: Proceedings of the Sixth International Research Conference on Fibonacci Numbers and their Applications, editors G E Bergum, A N Philippou, A F Horodam; Kluwer Academic (1996), pages 389 - 408.

S Vajda, "Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications", Halsted Press (1989).
This is a wonderful book, a classic but now unfortunately out of print. Vajda packs the book full of formulae on the Fibonacci numbers and Phi and the Lucas numbers. The whole book develops these formulae step by step, proving each from earlier ones or occasionally from scratch. It has a few errors in its formulae and all of them have been dutifully and exactly copied by authors such as Dunlap above! Where I have identified errors, they have been corrected above.

Fibonacci and Phi in the Arts	Fibonacci Home Page This is the first page on this Topic. **WHERE TO NOW???** Links and Bibliography	This is the last Topic. More topics at Ron Knott's Home page