Fibonacci and Golden Ratio Formulae

Here are almost 300 formula involving the Fibonacci numbers and the golden ratio together with the Lucas numbers and the General Fibonacci series (the G series). This forms a major reference page for Ron Knott's Fibonacci Web site (http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/fib.html) where there are many more details and explanations with applications, puzzles and investigations aimed at secondary school students and teachers as well as interested mathematical enthusiasts.
Note that it is **easy to search for a named formula** on this page since it is an HTML page and the formulae are not images. In your browser main menu, under the **Edit** menu look for **Find...** and type Vajda-*N* or Dunlap-*N* for the relevant formula. Full references are at the foot of this document.

A companion page on Linear Recurrences and their generating Functions for Fibonacci Numbers, Continued Fraction convergents, Pythagorean triples and other series of numbers.

Contents of this page

1 Definitions and Notation

Beware of different golden ratio symbols used by different authors!
Where a formula below (or a simple re-arrangement of it) occurs in either Vajda or Dunlap's book, the reference number they use is given here. Dunlap's formulae are listed in his Appendix A3. Hoggatt's formula are from his "Fibonacci and Lucas Numbers" booklet. Full bibliographic details are at the end of this page in the References section.

As used here	**Vajda**	**Dunlap**	**Knuth**	**Definition**	**Description**
Phi Φ	τ	τ	ϕ, α	$\frac{\sqrt{5}+1}{2}$ = 1.6180339...	*Koshy uses* α *(page 78)*
phi ϕ	$-\sigma$	$-\phi$	$-\beta$	$\frac{\sqrt{5}-1}{2}$ = 0.6180339...	*Koshy uses* $-\beta$ *(page 78)*
abs(x) $\lvert x\rvert$	$\lvert x\rvert$	$\lvert x\rvert$	$\lvert x\rvert$	*abs(x) = x if x≥0; abs(x) = –x if x<0*	*the absolute value of a number, its magnitude; ignore the sign;*
floor(x) $\lfloor x\rfloor$	*[x]*	*trunc(x), not used for x<0*	$\lfloor x\rfloor$	*the nearest integer ≤ x.*	***When x>0**, this is "the integer part of x" or "truncate x" i.e. delete any fractional part after the decimal point. 3=floor(3)=floor(3.1)=floor(3.9), -4=floor(-4)=floor(-3.1)=floor(-3.9)*
round(x) [*x*]	$\left[x+\frac{1}{2}\right]$	*trunc(x + 1/2)*		*the nearest integer to x; trunc(x+0.5)*	*3=round(3)=round(3.1), 4=round(3.9), -4=round(-4)=round(-3.9), -3=round(-3.1) 4=round(3.5), -3=round(-3.5)*
ceil(x) $\lceil x\rceil$	-	-	$\lceil x\rceil$	*the nearest integer ≥ x.*	*3=ceil(3), 4=ceil(3.1)=ceil(3.9), -3=ceil(-3)=ceil(-3.1)=ceil(-3.9)*
fract(x) frac(x)	-	-	***x mod 1***	*x – floor(x)*	*the fractional part of x, i.e. the part of abs(x)* ***after*** *the decimal point*
$\binom{n}{r}$	$\binom{n}{r}$	$\binom{n}{r}$	$\binom{n}{r}$	$\frac{n!}{r!\,(n-r)!}$	${}_nC_r$ *n choose r;* *the element in row n column r of Pascal's Triangle* *the coefficient of* x^r *in* $(1+x)^n$ *the number of ways of choosing r objects from a set of n different objects.* *n≥0 and r≥0 (otherwise value is 0)*

Fibonacci-type series with the rule S(i)=S(i-1)+S(i-2) for all integers i:

i	...	–6	–5	–4	–3	–2	–1	0	1	2	3	4	5	6	...
Fibonacci F(i)	...	–8	5	–3	2	–1	1	0	1	1	2	3	5	8	...
Lucas L(i)	...	18	–11	7	–4	3	–1	2	1	3	4	7	11	18	...
General Fib G(a,b,i)	...	13a–8b	–8a+5b	5a–3b	–3a+2b	2a–b	–a+b	**a**	**b**	a+b	a+2b	2a+3b	3a+5b	5a+8b	...

Formula	***Refs***	*Comments*
$F(0) = 0$, $F(1) = 1$, $F(n+2) = F(n+1) + F(n)$	-	Definition of the Fibonacci series
$F(-n) = (-1)^{n+1} F(n)$	Vajda-2, Dunlap-5	Extending the Fibonacci series 'backwards'
$L(0) = 2$, $L(1) = 1$, $L(n+2) = L(n+1) + L(n)$	-	Definition of the Lucas series

$L(-n) = (-1)^n L(n)$	Vajda-4, Dunlap-6	Extending the Lucas series 'backwards'
$G(n+2) = G(n+1) + G(n)$	Vajda-3, Dunlap-4	Definition of the Generalised Fibonacci series, G(0) and G(1) needed
$Phi = 1.618... = \frac{\sqrt{5}+1}{2}$	Dunlap-63	Phi and –phi are the roots of $x^2 = x + 1$
$phi = 0.618... = \frac{\sqrt{5}-1}{2}$	Dunlap-65	Beware! Dunlap occasionally uses ϕ to represent our phi = 0.61803.., but more frequently he uses ϕ to represent –0.61803.. !

2 Linear Formulae

Linear relationships involve only sums or differences of Fibonacci numbers or Lucas numbers or their multiples.

2.1 Linear Sums of Fibonacci numbers

$F(n+2) + F(n) + F(n-2) = 4\,F(n)$	B&Q(2003)-Identity 18
$F(n+2) + F(n) = L(n+1)$	by Definition of L(n), Vajda-6, Hoggatt-I8, Dunlap-14, Koshy-5.14
$F(n+2) - F(n) = F(n+1)$	by Definition of F(n)
$F(n+3) + F(n) = 2\,F(n+2)$	B&Q(2003)-Identity 16
$F(n+3) - F(n) = 2\,F(n+1)$	-
$F(n+4) + F(n) = 3\,F(n+2)$	B&Q(2003)-Identity 17
$F(n+2) + F(n-2) = 3\,F(n)$	B&Q(2003)-Identity 7
$F(n+2) - F(n-2) = L(n)$	Hoggatt-I14
$F(n+4) - F(n) = L(n+2)$	-
$F(n+5) + F(n) = F(n+2) + L(n+3)$	-
$F(n+5) - F(n) = L(n+2) + F(n+3)$	-
$F(n+6) + F(n) = 2\,L(n+3)$	-
$F(n+6) - F(n) = 4\,F(n+3)$	-
$F(n) + 2\,F(n-1) = L(n)$	(Dunlap-32)
$F(n+2) - F(n-2) = L(n)$	Vajda-7a, Dunlap-15, Koshy-5.15
$F(n+3) - 2\,F(n) = L(n)$	possible correction for Dunlap-31
$F(n+2) - F(n) + F(n-1) = L(n)$	possible correction for Dunlap-31
$F(n) + F(n+1) + F(n+2) + F(n+3) = L(n+3)$	C Hyson(*)

2.2 Linear Sums of Lucas numbers

$L(n-1) + L(n+1) = 5\,F(n)$	Vajda-5, Dunlap-13, Koshy-5.16, B&Q(2003)-Identity 34, Hoggatt-I9
$L(n) + L(n+3) = 2\,L(n+2)$	-
$L(n) + L(n+4) = 3\,L(n+2)$	-
$2\,L(n) + L(n+1) = 5\,F(n+1)$	B&Q(2003)-Identity 52
$L(n+2) - L(n-2) = 5\,F(n)$	-
$L(n+3) - 2\,L(n) = 5\,F(n)$	-

2.3 Linear Sum of a Fibonacci and a Lucas number

$F(n) + L(n) = 2\ F(n+1)$	Vajda-7b, Dunlap-16, B&Q-Identity 51
$L(n) + 5\ F(n) = 2\ L(n+1)$	-
$3\ F(n) + L(n) = 2\ F(n+2)$	Vajda-26, Dunlap-28
$3\ L(n) + 5\ F(n) = 2\ L(n+2)$	Vajda-27, Dunlap-29

2.4 Golden Ratio Formulae

$$Phi = \frac{\sqrt{5}+1}{2};\ phi = \frac{\sqrt{5}-1}{2}$$

2.4.1 Basic Phi Formulae

Defining relations: The roots of $x^2 = x + 1$ are

$$Phi = \frac{1+\sqrt{5}}{2} = 1.6180339887498948482045868343\ 6...$$

$$-phi = \frac{1-\sqrt{5}}{2} = -0.6180339887498948482045868343\ 6...$$

$Phi\ phi = 1$	Vajda page 51(3), Dunlap-65
$Phi + phi = \sqrt{5}$	-
$Phi\ /\ phi = Phi + 1$	-
$phi\ /\ Phi = 1 - phi$	-
$Phi = phi + 1 = \sqrt{5} - phi$	-
$phi = Phi - 1 = \sqrt{5} - Phi$	-
$Phi^2 = Phi + 1$	Vajda page 51(4), Dunlap-64
$Phi^{n+2} = Phi^{n+1} + Phi^n\ \forall n \in \mathbb{Z}$	$Phi^n \times$ Vajda page 51(4)
$phi^2 = 1 - phi$	Vajda page 51(4), Dunlap-64
$phi^{n+2} = phi^n - phi^{n+1}\ \forall n \in \mathbb{Z}$	$phi^n \times$Vajda page 51(4)
$phi^n = phi^{n+1} + phi^{n+2}\ \forall n \in \mathbb{Z}$	from line above

2.4.2 Golden Ratio with Fibonacci and Lucas - Exact

$F(n) = \frac{Phi^n - (-phi)^n}{\sqrt{5}}$	**"Binet's" Formula** De Moivre(1718), Binet(1843), Lamé(1844), Vajda-58, Dunlap-69, Hoggatt-page 11, B&Q(2003)-Identity 240
$L(n) = Phi^n + (-phi)^n$	Vajda-59, Dunlap-70, B&Q(2003)-Identity 241
$Phi^n = Phi\ F(n) + F(n-1)$	Vajda page 52 (Vajda-50a), Rabinowitz-28, B&Q(2003)-Corrolary 33
$Phi^n = F(n+1) + F(n)\ phi$	Rabinowitz-28, B&Q(2003)-Corollary 33
$Phi^n = \frac{L(n) + F(n)\sqrt{5}}{2}$	Vajda page 52 (Vajda-50b), Rabinowitz-25, B&Q(2003)-Identity 242, I Ruggles (1963) FQ 1.2 pg 80
$(-phi)^n = \frac{L(n) - F(n)\sqrt{5}}{2}$	Vajda page 52 (Vajda-50c), I Ruggles (1963) FQ 1.2 pg 80, Rabinowitz-25, B&Q(2003)-Identity 243
$(-phi)^n = -phi\ F(n) + F(n-1)$	Rabinowitz-28

$(-phi)^n = F(n+1) - Phi\ F(n)$	Vajda-103b, Dunlap-75
$\sqrt{5}\ Phi^n = Phi\ L(n) + L(n-1)$	-
$\sqrt{5}\ (-phi)^n = phi\ L(n) - L(n-1)$	-

2.4.3 Golden Ratio with Fibonacci and Lucas - Approximations

$\lim_{n\to\infty} \frac{F(n+1)}{F(n)} = Phi$	Vajda-101
$\lim_{n\to\infty} \frac{F(n+m)}{F(n)} = Phi^m$	Vajda-101a
$F(n) = round\left(\frac{Phi^n}{\sqrt{5}}\right), if\ n\geq 0$	Vajda-62, Dunlap-71 corrected, B&Q(2003)-Identity 240 Corollary 30
$L(n) = round(Phi^n), if\ n\geq 2$	Vajda-63, Dunlap-72, B&Q(2003)-Corollary 35
$F(-n) = round\left(\frac{-(-phi)^{-n}}{\sqrt{5}}\right), if\ n\geq 0$	-
$L(-n) = round(\ (-phi)^{-n}\ ),\ n\geq 2$	-
$F(n+1) = round(Phi\ F(n)), if\ n\geq 2$	Vajda-64, Dunlap-73
$L(n+1) = round(Phi\ L(n)), if\ n\geq 4$	Vajda-65, Dunlap-74
$fract(\ F(2n)\ phi\ ) = 1 - phi^{2n}$	Knuth vol 1, Ex 1.2.8 Qu 31 with ψ=phi
$fract(\ F(2n+1)\ phi\ ) = phi^{2n+1}$	Knuth vol 1, Ex 1.2.8 Qu 31

3 Fibonacci and Lucas Factors

$F(nk)$ is a multiple of $F(n)$ $F(nk) \equiv 0 \pmod{F(k)}$	B&Q(2003)-Theorem 1, Vajda Theorem I page 82
$gcd(F(m),F(n)) = F(gcd(m,n))$	Lucas (1878) B&Q(2003)-Theorem 6,Vajda Theorem II page 83
$F(mn+r) \equiv \pm F(r) \pmod{F(n)}$	Knuth Vol 1 Ex 1.2.8 Qu. 32, Vajda page 86
$gcd(L(m),L(n)) = L(gcd(m,n))$, if both $L(m)/gcd(L(m),L(n))$ and $L(n)/gcd(L(m),L(n))$ are odd integers	Vajda page 86
$L(mn+r) \equiv \pm L(r) \pmod{L(n)}$	(Vajda page 87)
$F(m\ q) = F(m) \sum_{j=1}^{q} F(m-1)^{j-1} F(\ m(q-j)+1\ )$	B&Q(2003)-Theorem 2
$\frac{F(kt)}{F(t)} = \sum_{i=0}^{(k-3)/2} (-1)^{it} L(\ (k-2i-1)t\ ) + (-1)^{(k-1)t/2}$ for ODD $k \geq 3$	Vajda-85
$\frac{F(kt)}{F(t)} = \sum_{i=0}^{k/2-1} (-1)^{it} L(\ (k-2i-1)t\ )$ for EVEN $k \geq 2$	Vajda-86
$\frac{L(kt)}{L(t)} = \sum_{i=0}^{(k-3)/2} (-1)^{i(t+1)} L(\ (k-2i-1)t\ ) + (-1)^{(k-1)(t+1)/2}$ for ODD $k \geq 3$	Vajda-87

L(t) is not a factor of L(kt) for even k

$$\frac{F(kt)}{L(t)} = \sum_{i=0}^{k/2-1} (-1)^{i(t+1)} F(\,(k-2i-1)t\,) \quad \text{for EVEN } k \ge 2$$ Vajda-88

L(t) is not a factor of F(kt) for odd k and t≥3

4 Order 2 Formulae

Order 2 means these formulae have terms involving the *product of at most 2* Fibonacci or Lucas numbers.

4.1 Fibonacci numbers

$F(n)^2 + 2\,F(n-1)F(n) = F(2n)$	-
$F(n+1)^2 + F(n)^2 = F(2n+1)$	Vajda-11, Dunlap-7, Lucas(1878), B&Q(2003)-Identity 13, Hoggatt-I11
$F(n+1)^2 - F(n-1)^2 = F(2n)$	Lucas(1878), B&Q(2003)-Identity 14, Hoggatt-I10
$F(n+1)^2 - F(n)^2 = F(n+2)\,F(n-1)$	Vajda-12, Dunlap-8
$F(n+2)^2 = 3\,F(n+1)^2 - F(n)^2 - 2\,(-1)^n$	V E Hoggatt B-208 FQ 9 (1971) pg 217.
$F(n+3)^2 + F(n)^2 = 2\,(\,F(n+1)^2 + F(n+2)^2\,)$	B&Q(2003)-Identity 30
$F(n+k+1)^2 + F(n-k)^2 = F(2k+1)F(2n+1)$	Sharpe(1965), a generalization of Vajda-11,Dunlap-7 Melham(1999)
$F(n+k)^2 + F(n-k)^2 = F(n+k-2)F(n+k+1) + F(2k-1)F(2n-1)$	Sharpe (1965)
$F(n+k+1)^2 - F(n-k)^2 = F(n-k-1)F(n-k+2) + F(2k)F(2n+2)$	Sharpe (1965)
$F(\,n+p\,)^2 - F(\,n-p\,)^2 = F(\,2n\,)F(\,2p\,)$	I Ruggles (1963) FQ 1.2 pg 77; Hoggatt-I25, Sharpe (1965)
$F(n+1)\,F(n-1) - F(n)^2 = (-1)^n$	**Cassini's Formula**(1680), Simson(1753), Vajda-29, Dunlap-9, Hoggatt-I13 special case of Catalan's Identity with r=1 B&Q(2003)-Identity 8
$F(n)^2 - F(n+r)F(n-r) = (-1)^{n-r}F(r)^2$	**Catalan's Identity**(1879)
$F(n)F(m+1) - F(m)F(n+1) = (-1)^m F(n-m)$	**d'Ocagne's Identity**, special case of Vajda-9 with G=F
$F(n+m) = F(n+1)F(m+1) - F(n-1)F(m-1)$	B&Q(2003)-Identity 231
$F(n+m) = F(m)\,F(n+1) + F(m-1)\,F(n)$	alternative to Dunlap-10, B&Q(2003)-Identity 3; variation of Hansen (1972)
$F(n) = F(m)\,F(n+1-m) + F(m-1)\,F(n-m)$	I Ruggles (1963) FQ 1.2 pg 79; Dunlap-10, special case of Vajda-8
$F(n)\,F(n+1) = F(n-1)\,F(n+2) + (-1)^{n-1}$	Vajda-20a special case: i:=1;k:=2;n:=n-1; Hoggatt-I19
$F(n+i)\,F(n+k) - F(n)\,F(n+i+k) = (-1)^n\,F(i)\,F(k)$	Vajda-20a=Vajda-18 (corrected) with G:=H:=F
$2\,F(n+1) = F(n) + \surd(5\,F(n)^2 + 4(-1)^n)$	F(n+1) from F(n): Problem B-42, S Basin, FQ, 2 (1964) page 329
$F(a)F(b) - F(c)F(d)$ $= (-1)^r(\,F(a-r)F(b-r) - F(c-r)F(d-r)\,)$ *a+b=c+d for any integers a,b,c,d,r*	Johnson FQ 42 (2004) B-960 'A Fibonacci Iddentity', solution pg 90 also Johnson-7 Cassini, Catalan and D'Ocagne's Identities are all special cases of this formula

4.2 Lucas numbers

$L(n+2)^2 = 3\ L(n+1)^2 - L(n)^2 + 10(-1)^n$	V E Hoggatt B-208 FQ 9 (1971) pg 217.
$L(n+2)\ L(n-1) = L(n+1)^2 - L(n)^2$	-
$L(n+1)\ L(n-1) - L(n)^2 = -5\ (-1)^n$	B&Q(2003)-Identity 60
$L(2n) + 2\ (-1)^n = L(n)^2$	Vajda-17c, Dunlap-12, B&Q(2003)-Identity 36
$L(n+m) + (-1)^m\ L(n-m) = L(m)\ L(n)$	Vajda-17a, Dunlap-11 (special cases: Hoggatt-I15,I18)
$L(4n) + 2 = L(2n)^2$	Hoggatt-I15, special case of Vajda-17a
$2\ L(n+1) = L(n) + \sqrt{5}\ \sqrt{(L(n)^2 - 4(-1)^n)}$	L(n+1) from L(n): Problem B-42, S Basin, FQ 2 (1964) page 329

4.3 Fibonacci and Lucas Numbers

$F(2n) = F(n)\ L(n)$	Vajda-13, Hoggatt-I7, Koshy-5.13, B&Q(2003)-Identity 33
$5\ F(n) = L(n+1) + L(n-1)$	
$L(n+1)^2 + L(n)^2 = 5\ F(2n+1)$	Vajda-25a
$L(n+1)^2 - L(n-1)^2 = 5\ F(2n)$	
$L(n+1)^2 - 5\ F(n)^2 = L(2n+1)$	
$L(2n) - 2\ (-1)^n = 5\ F(n)^2$	Vajda-23, Dunlap-25
$L(n)^2 - 4(-1)^n = 5\ F(n)^2$	B&Q(2003)-Identity 53, Hoggatt-I12
$F(n+k) + F(n-k) = F(n)L(k)$, k even;	Bergum and Hoggatt (1975) equn (5)
$F(n+k) + F(n-k) = L(n)F(k)$, k odd;	Bergum and Hoggatt (1975) equn (6)
$F(n+k) - F(n-k) = F(n)L(k)$, k odd;	Bergum and Hoggatt (1975) equn (7)
$F(n+k) - F(n-k) = L(n)F(k)$, k even;	Bergum and Hoggatt (1975) equn (8)
$L(n+k) + L(n-k) = L(n)L(k)$, k even	Bergum and Hoggatt (1975) equn (9)
$L(n+k) + L(n-k) = 5F(n)F(k)$, k odd	Bergum and Hoggatt (1975) equn (10)
$L(n+k) - L(n-k) = L(n)L(k)$, k odd	Bergum and Hoggatt (1975) equn (11)
$L(n+k) - L(n-k) = 5F(n)F(k)$, k even	Bergum and Hoggatt (1975) equn (12)
$F(n+1)\ L(n) = F(2n+1) + (-1)^n$	Vajda-30, Vajda-31, Dunlap-27, Dunlap-30
$L(n+1)\ F(n) = F(2n+1) - (-1)^n$	-
$F(2n+1) = F(n+1)\ L(n+1) - F(n)\ L(n)$	Vajda-14, Dunlap-18
$L(2n+1) = F(n+1)\ L(n+1) + F(n)\ L(n)$	-
$L(m)\ L(n) + L(m-1)\ L(n-1) = 5\ F(m+n-1)$	Hansen 1972
$L(n)^2 - 2\ L(2n) = -5\ F(n)^2$	Vajda-22, Dunlap-24
$5\ F(n)^2 - L(n)^2 = 4\ (-1)^{n+1}$	Vajda-24, Dunlap-26
$F(n)^2 + L(n)^2 = 4\ F(n+1)^2 - 2\ F(2n)$	FQ (2003)vol 41, B-936, M A Rose, page 87
$5\ (F(n)^2 + F(n+1)^2) = L(n)^2 + L(n+1)^2$	Vajda-25

$F(n)\,L(m) = F(n+m) + (-1)^m\,F(n-m)$	a recurrence relation for F(n+km): Vajda-15a, Dunlap-19
$L(n)\,F(m) = F(n+m) - (-1)^m\,F(n-m)$	Vajda-15b, Dunlap-20
$5\,F(m)\,F(n) = L(n+m) - (-1)^m\,L(n-m)$	Vajda-17b, Dunlap-23, (special cases:Hoggatt-I16,I17)
$2\,F(n+m) = L(m)\,F(n) + L(n)\,F(m)$	Vajda-16a, Dunlap-2, FQ (1967) B106 H H Ferns pp 466-467
$2\,L(n+m) = L(m)\,L(n) + 5\,F(n)\,F(m)$	FQ (1967) B106 H H Ferns pp 466-467
$F(m)\,L(n) + F(m-1)\,L(n-1) = L(m+n-1)$	Hansen (1972)
$(-1)^m\,2\,F(n-m) = L(m)\,F(n) - L(n)\,F(m)$	Vajda-16b, Dunlap-22
$L(n+i)\,F(n+k) - L(n)\,F(n+i+k) =$ $(-1)^{n+1}\,F(i)\,L(k)$	Vajda-19a
$F(n+i)\,L(n+k) - F(n)\,L(n+i+k) = (-1)^n\,F(i)\,L(k)$	Vajda-19b
$L(n+k+1)^2 + L(n-k)^2 = 5\,F(2n+1)F(2k+1)$	Melham (1999) Theorem 1
$L(n+i)\,L(n+k) - L(n)\,L(n+i+k)$ $= (-1)^{n+1}\,5\,F(i)\,F(k)$	Vajda-20b
$(-1)^k F(n)F(m-k) + (-1)^m F(k)F(n-m) + (-1)^n F(m)F(k-n) = 0$	FQ 11 (1973) B228 page 108
$(-1)^k L(n)F(m-k) + (-1)^m L(k)F(n-m) + (-1)^n L(m)F(k-n) = 0$	FQ 11 (1973) B229 page 108
$5\,F(jk+r)\,F(ju+v) = L(j(k+u)+(r+v)) - (-1)^{ju+v} L(j(k-u)+(r-v))$	Hansen (1978)
$F(jk+r)\,L(ju+v) = F(j(k+u)+(r+v)) + (-1)^{ju+v} F(j(k-u)+(r-v))$	Hansen (1978)
$L(jk+r)\,L(ju+v) = L(j(k+u)+(r+v)) + (-1)^{ju+v} L(j(k-u)+(r-v))$	Hansen (1978)
$5F(a)F(b) - L(c)L(d) = (-1)^r(\,5F(a-r)F(b-r) - L(c-r)L(d-r)\,)$ *a+b=c+d for any integers a,b,c,d,r*	Johnson
$F(a)\,L(b) - F(c)\,L(d) = (-1)^r(\,F(a-r)\,L(b-r) - F(c-r)\,L(d-r)$ *with a+b=c+d*	Johnson-32, special case of Johnson-44

5 Higher Order Fibonacci and Lucas

5.1 Fibonacci and Lucas cubed

$F(3n) = F(n+1)^3 + F(n)^3 - F(n-1)^3$	Lucas (1876), B&Q(2003)-Identity 232
$5\,L(3n) = L(n+1)^3 + L(n)^3 - 3\,L(n-1)^3$	Long (1986) equation (45)
$3\,F(3n) = F(n+2)^3 - 3\,F(n)^3 + F(n-2)^3$	J Ginsburg "A Relationship Between Cubes of Fibonacci Numbers." *Scripta Mathematica* (1953) page 242.
$L(3n) = L(n+1)F(n+1)^2 + L(n)F(n)^2 - L(n-2)F(n-1)^2)$	Long (1986) equation (43)
$5\,F(3n) = F(n+1)L(n+1)^2 + F(n)L(n)^2 - F(n-1)L(n-1)^2$	Long (1986) equation (44)
$F(n+1)F(n+2)F(n+6) - F(n+3)^3 = (-1)^n F(n)$ $F(n)F(n+4)F(n+5) - F(n+3)^3 = (-1)^{n+1} F(n+6)$	FQ 41 (2003) pg 142, Melham. The second is a variant with -n for n and using Vajda-2
$F(n-2)F(n-1)F(n+3) - F(n)^3 = (-1)^{n-1} F(n-3)$ $F(n+2)F(n+1)F(n-3) - F(n)^3 = (-1)^n F(n+3)$	Fairgrieve and Gould (2005) versions of the above two formulae of Melham
$F(n-2)F(n+1)^2 - F(n)^3 = (-1)^{n-1}\,F(n-1)$ $F(n+2)F(n-1)^2 - F(n)^3 = (-1)^n\,F(n+1)$	Fairgrieve and Gould (2005)

$F(n+a+b)F(n-a)F(n-b) - F(n-a-b)F(n+a)F(n+b) = (-1)^{n+a+b}F(a)F(b)F(a+b)L(n)$	Melham (2011) Theorem 1
$F(n+a+b-c)F(n-a+c)F(n-b+c) - F(n-a-b+c)F(n+a)F(n+b) = (-1)^{n+a+b+c}F(a+b-c)(F(c)F(n+a+b-c) + (-1)^c F(a-c)F(b-c)L(n))$	Melham (2011) Theorem 5
$F(i+j+k) = F(i+1)F(j+1)F(k+1) + F(i)F(j)F(k) - F(i-1)F(j-1)F(k-1)$ for any integers i,j,k	Johnson's (6)
$L(5n) = L(n) (L(2n) + 5F(n) + 3)(L(2n) - 5F(n) + 3)$, n odd	**Aurifeuille's Identity (1879)** FQ 42 (2004) R S Melham, pgs 155-160

5.2 Fibonacci and Lucas to the fourth

$F(n-1)^2F(n+1)^2 - F(n-2)^2F(n+2)^2 = 4(-1)^nF(n)^2$	Melham (2011) 21
$F(n-3)F(n-1)F(n+1)F(n+3) - F(n)^4 = (-1)^nL(n)^2$	Melham (2011) 22
$F(n)^2 F(m+1) F(m-1) - F(m)^2 F(n+1) F(n-1) = (-1)^{n-1} F(m+n) F(m-n)$	Vajda-32
$F(n-2)F(n-1)F(n+1)F(n+2) + 1 = F(n)^4$	**Gelin-Cesàro Identity** (1880) (see Dickson page 401) FQ 41 (2003) pg 142, B&Q(2003)-Identity 31 Hoggatt-I29, Simson(1753)
$L(n-2)L(n-1)L(n+1)L(n+2) + 25 = L(n)^4$	B&Q(2003)-Identity 56
$F(n+a+b+c)F(n-a)F(n-b)F(n-c) - F(n-a-b-c)F(n+a)F(n+b)F(n+c) = (-1)^{n+a+b+c}F(a+b)F(a+c)F(b+c)F(2n)$	Melham (2011) Theorem 2
$F(n+a+b+c-d)F(n-a+d)F(n-b+d)F(n-c+d) - F(n-a-b-c+2d)F(n+a)F(n+b)F(n+c) = (-1)^{n+a+b+c}F(a+b-d)F(a+c-d)F(b+c-d)F(2n+d)$	Melham (2011) Theorem 6
$(F(n-1)F(n+2))^2 + (2 F(n)F(n+1))^2 = (F(n+1)F(n+2) - F(n-1)F(n))^2 = F(2n+1)^2$	A F Horadam *FQ* 20 (1982) pgs 121-122, B&Q(2003)-Identity 19 (corrected) special case of **Generalised Fibonacci Pythagorean Triples**
$(F(n)^2 + F(n+1)^2 + F(n+2)^2)^2 = 2 (F(n)^4 + F(n+1)^4 + F(n+2)^4)$	**Candido's Identity (1951)** FQ 42 (2004) R S Melham, pgs 155-160
$(L(n-1)L(n+2))^2 + (2L(n)L(n+1))^2 = (5F(2n+1))^2$	Wulczyn *FQ* 18 (1980) pg 188 special case of **Generalised Fibonacci Pythagorean Triples**

5.3 Fibonacci and Lucas Higher Powers

$F(n)F(n+1)F(n+2)F(n+4)F(n+5)F(n+6) + L(n+3)^2 = (F(n+3)(2F(n+2)F(n+4) - F(n+3)^2))^2$	Morgado (1987)
$\left(\frac{L(n) + \sqrt{5}\, F(n)}{2}\right)^k = \frac{L(kn) + \sqrt{5}\, F(kn)}{2}$	**De Moivre Analogue**, S Fisk (1963) FQ 1.2 Problem B-10, pg 85. Hoggatt-I44

6 Fibonomial formulae

The Fibonomials are defined using Fibonacci numbers instead of integers in binomial coefficients and Fibonacci factorials instead of normal factorials. There are many analagous results to those using binomial coefficients but using Fibonomials instead.

We define $F!(n) = F(n)F(n-1)...F(2)F(1), n>0; F!(0)=1$ for which some authors use $n!_F$, to compare with $n! = n(n-1)...3.2.1$.
There is no universal notation for the Fibonomial. The fibonomial "Fibonacci n choose k" is defined as:

$$\binom{n}{k}_F = \frac{F!(n)}{F!(k)\,F!(n-k)} = \frac{F(n)F(n-1)...F(n-k+1)\;F(n-k)F(n-k-1)...F(2)F(1)}{F(k)F(k-1)...F(2)F(1)\;\;F(n-k)F(n-k-1)...F(2)F(1)} \quad if\ n \ge k \ge 0$$
$$= 0, otherwise$$

Vajda (page 74) uses $J(n,k)$. D Knuth and others use double brackets: $\left(\!\binom{n}{k}\!\right)$ while Melham (1999) and others use square brackets: $\begin{bmatrix} n \\ k \end{bmatrix}$
A simple alternative is to write $fibonomial(n,k)$.
Here is a table of some values of the fibonomial (A010048)

n \ k	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	1	1					
3	1	2	2	1				
4	1	3	6	3	1			
5	1	5	15	15	5	1		
6	1	8	40	60	40	8	1	
7	1	13	104	260	260	104	13	1

$$\binom{m+n}{n}_F = F(m-1)\binom{m+n-1}{n-1}_F + F(n+1)\binom{m+n-1}{n}_F$$

Vajda page 74,
"add the two numbers above" analogy from Pascal's triangle

$$\sum_{j=0}^{m} (-1)^{j(j+3)/2} \binom{m}{j}_F F(n+m-j)^{m+1} = F!(m)\; F((m+1)n+m(m+1)\,/2))$$

Melham (1999)....

$\mathbf{1}\,F(n+1)^2 + \mathbf{1}\,F(n)^2$	$= 1$	$F(2n+1)$
$\mathbf{1}\,F(n+2)^3 + \mathbf{1}\,F(n+1)^3 - \mathbf{1}\,F(n)^3$	$= 1.1$	$F(3n+3)$
$\mathbf{1}\,F(n+3)^4 + \mathbf{2}\,F(n+2)^4 - \mathbf{2}\,F(n+1)^4 - \mathbf{1}\,F(n)^4$	$= 1.1.2$	$F(4n+6)$
$\mathbf{1}\,F(n+4)^5 + \mathbf{3}\,F(n+3)^5 - \mathbf{6}\,F(n+2)^5 - \mathbf{3}\,F(n+1)^5 + \mathbf{1}\,F(n)^5$	$= 1.1.2.3$	$F(5n+10)$
$\mathbf{1}\,F(n+5)^6 + \mathbf{5}\,F(n+4)^6 - \mathbf{15}\,F(n+3)^6 - \mathbf{15}\,F(n+2)^6 + \mathbf{5}\,F(n+1)^6 + \mathbf{1}\,F(n)^6$	$= 1.1.2.3.5$	$F(6n+15)$

.... examples

$0 = F(n) - F(n-1) - F(n-2)$
$0 = F(n)^2 - 2\,F(n-1)^2 - 2\,F(n-2)^2 + F(n-3)^2$
$0 = F(n)^3 - 3\,F(n-1)^3 - 6\,F(n-2)^3 + 3\,F(n-3)^3 + F(n-4)^3$
$0 = F(n)^4 - 5\,F(n-1)^4 - 15\,F(n-2)^4 + 15\,F(n-3)^4 + 5\,F(n-4)^4 - F(n-5)^4$
...

Brousseau (1968)...but the general formula was not given.
For this see next line:

$$\sum_{k=0}^{p} \binom{p}{k}_F (-1)^{\lceil k/2 \rceil} F(n-k)^{p-1} = 0,\ if\ p>0$$

Knuth AoCP Vol 1 section 1.2.8 Exercise 30, (1997)

$F(k) \binom{n}{k}_F = F(n-k+1) \binom{n}{k-1}_F$ compare with $\binom{n}{k} = \frac{n-k+1}{k} \binom{n}{k-1}$

$F(k) \binom{n}{k}_F = F(n) \binom{n-1}{k-1}_F$ compare with $k \binom{n}{k} = n \binom{n-1}{k-1}$

$F(n-k) \binom{n}{k}_F = F(n) \binom{n-1}{k}_F$ compare with $n-k \binom{n}{k} = n \binom{n-1}{k}$

$\binom{n}{k}_F \binom{k}{j}_F = \binom{n}{j}_F \binom{n-j}{k-j}_F$ compare with $\binom{n}{k} \binom{k}{j} = \binom{n}{j} \binom{n-j}{k-j}$

7 G Formulae

G(i) is the General Fibonacci series. It has the same recurrence relation as Fibonacci and Lucas, namely **G(n+2) = G(n+1) + G(n) for all integers n (i.e. n can be negative)** Vajda-3,Dunlap 4, but the "starting values" of G(0)=a and G(1)=b can be specified. To make it clear which starting values for G(0)=a and G(1)=b are being used, we write G(a,b,i) for G(i). G(n) is an abbreviation for G(a,b,n) when a and b are understood from the context.
Special cases are the Fibonacci and Lucas series since F(n) = G(0,1,n) and L(n)=G(2,1,n):

- If G(0)=0 and G(1)=1 we have 0,1,1,2,3,5,8,13,.. the Fibonacci series, i.e. G(0,1,n) = F(n);
- G(0)=2 and G(1)=1 gives 2,1,3,4,7,11,18,.. the Lucas series, i.e. G(2,1,n) = L(n);

7.1 Basic G Formulae

Two independent G series are here denoted G(n) and H(n), i.e. G(0) and G(1) are independent of H(0) and H(1).

$G(n) = G(0)\ F(n-1) + G(1)\ F(n)$	B&Q(2003)-Identity 37
$G(-n) = (-1)^n\ (G(0)\ F(n+1) - G(1)\ F(n))$	ditto - applying Vajda-2
$\sqrt{5}\ G(n) = (\ G(0)\ phi + G(1)\ )\ Phi^n + (G(0)\ Phi - G(1))\ (\ -phi\ )^n$	Vajda-55/56, Dunlap-77, B&Q(2003)-Identity 244
$F(n) = \dfrac{G(0)\ G(n+1) - G(1)\ G(n)}{G(0)G(2) - G(1)^2}$	*Amer Math Monthly* (2005) "Fibonacci, Chebyshev and Orthogonal Polynomials" D Aharonov, A Beardam, K Driver, p612-630
$2\ G(k) = (\ 2\ G(1) - G(0)\ )\ F(k) + G(0)\ L(k)$	Johnson-46
$G(n+m) = F(m-1)\ G(n) + F(m)\ G(n+1)$	Vajda-8, Dunlap-33, B&Q(2003)-Identity 38, Johnson-40
$G(n-m) = (-1)^m\ (F(m+1)\ G(n) - F(m)\ G(n+1))$	Vajda-9, Dunlap-34, B&Q(2003)-Identity 47
$G(n+m) + (-1)^m\ G(n-m) = L(m)\ G(n)$	Vajda-10a, Dunlap-35, B&Q(2003)-Identity 45, Bergum & Hoggatt (1975) (36) and (38)
$G(n+m) - (-1)^m\ G(n-m) = F(m)\ (\ G(n-1) + G(n+1))$	Vajda-10b, Dunlap-36, B&Q(2003)-Identity 48, Bergum & Hoggatt (1975) (37) and (39)
$G(m)\ F(n) - G(n)\ F(m) = (-1)^{n+1}\ G(0)\ F(m-n)$	Vajda-21a
$G(m)\ F(n) - G(n)\ F(m) = (-1)^m\ G(0)\ F(n-m)$	Vajda-21b
$G(m+k)\ F(n+k) + (-1)^{k+1}\ G(m)\ F(n) = F(k)\ G(m+n+k)$	Howard(2003)

7.2 G Formulae of Order 2 or more

These formulae include terms which are a product of two G numbers either from the same G series of from two different G series i.e. with different index 0 and 1 values. Where the series may be different they are denoted G and H e.g. special cases include G = F (i.e. Fibonacci) and H = L (i.e. Lucas), or they could also be the same series G=H.

$G(n+i)\,H(n+k) - G(n)\,H(n+i+k)$ $= (-1)^n\,(G(i)\,H(k) - G(0)\,H(i+k))$	Vajda-18 (corrected), B&Q(2003)-Identity 44 a special case of Johnson's:
$G(p)H(q) - G(r)H(s)$ $= (-1)^n[\,G(p-n)H(q-n) - G(r-n)H(s-n)\,]$ *if p+q = r+s and p,q,r,s,n are integers*	Johnson-44
$G(n+1)\,G(n-1) - G(n)^2 = (-1)^n\,(G(1)^2 - G(0)\,G(2))$	Vajda-28, B&Q(2003)-Identity 46
$4\,G(n-1)G(n) + G(n-2)^2 = G(n+1)^2$	B&Q(2003)-Identity 65
$G(n+3)^2 + G(n)^2 = 2(\,G(n+1)^2 + G(n+2)^2\,)$	B&Q(2003)-Identity 70
$G(i+j+k) = F(i+1)F(j+1)G(k+1) + F(i)F(j)G(k) - F(i-1)F(j-1)G(k-1)$ *for any integers i,j,k*	Johnson (39a)
$4G(i)^2G(i+1)^2 + G(i-1)^2G(i+2)^2 = (\,G(i)^2 + G(i+1)^2\,)^2$	**Generalised Fibonacci Pythagorean Triples** Horadam (1967)
$G(n+2)G(n+1)G(n-1)G(n-2) + (\,G(2)G(0) - G(1)^2\,)^2$ $= G(n)^4$	B&Q(2003)-Identity 59

8 Summations

This section has formulae that sum a variable number of terms.

8.1 Fibonacci and Lucas Summations

These formulae involve a sum of Fibonacci or Lucas numbers only.

$\sum_{i=0}^{n} F(i) = F(n+2) - 1$	Hoggatt-I1, Lucas(1878), B&Q 2003-Identity 1
$\sum_{i=0}^{n} (-1)^i\,F(i) = (-1)^n\,F(n-1) - 1$	B&Q 2003-Identity 21
$\sum_{i=0}^{n} L(i) = L(n+2) - 1$	Hoggatt-I2
$\sum_{i=a}^{n} F(i) = F(n+2) - F(a+1)$	-
$\sum_{i=a}^{n} L(i) = L(n+2) - L(a+1)$	-
$\sum_{i=0}^{n} F(2i) = F(2n+1) - 1,\ n\geq 0$	Hoggatt-I6, Lucas(1878), B&Q(2003)-Identity 12
$\sum_{i=1}^{n} F(2i-1) = F(2n),\ n\geq 1$	Hoggatt-I5, Lucas(1878), B&Q(2003)-Identity 2
$\sum_{i=1}^{n} L(2i-1) = L(2n) - 2$	-

$$\sum_{i=1}^{n} 2^{n-i} F(i-1) = 2^n - F(n+2)$$ Vajda-37a(adapted), Dunlap-42(adapted), B&Q(2003)-Identity 10

$$\sum_{i=0}^{n} 2^i L(i) = 2^{n+1} F(n+1)$$ B&Q(2003)-Identity 236

$$\sum_{i=0}^{n} F(3i+1) = \frac{F(3n+3)}{2}$$ B&Q(2003)-Identity 23

$$\sum_{i=0}^{n} F(3i+2) = \frac{F(3n+4)-1}{2}$$ B&Q(2003)-Identity 24 (corrected)

$$\sum_{i=0}^{n} F(3i) = \frac{F(3n+2)-1}{2}$$ B&Q(2003)-Identity 25 (corrected)

$$\sum_{i=0}^{n} F(4i) = F(2n+1)^2 - 1$$ B&Q 2003-Identity 27

$$\sum_{i=0}^{n} F(4i+1) = F(2n+1)F(2n+2)$$ B&Q 2003-Identity 26

$$\sum_{i=0}^{n} F(4i+2) = F(2n+1)F(2n+3) - 1$$ B&Q 2003-Identity 29

$$\sum_{i=0}^{n} F(4\,i+3) = F(2n+3)F(2n+2)$$ B&Q 2003-Identity 28

$$\sum_{i=0}^{n} (-1)^i L(n-2i) = 2\,F(n+1)$$ Vajda-97, Dunlap-54

$$\sum_{i=0}^{n} (-1)^i L(2n-2i+1) = F(2\,n+2)$$ B&Q(2003)-Identity 55

8.2 Decimal (and other bases) fractions

We saw in The Fibonacci Series as a Decimal Fraction that the Fibonacci series occurs naturally as the decimal expansion of a simple fraction in several ways:

$1/89 = 0.0\ 1\ 1\ 2\ 3\ 5\ ...$

$1/9899 = 0.00\ 01\ 01\ 02\ 03\ 05\ 08\ 13\ 21\ ...$

with a varying number of decimal digits before the Fibonacci numbers overlap and the series is obscured. This section gives formulae for these fractions for various subsequences of Fibonacci and General Fibonacci series.

$$\sum_{k=1}^{\infty} 10^{-n(k+1)} F(ak) = \frac{F(a)}{10^{2n} - 10^n L(a) - (-1)^a}$$ Hudson and Winans (1981)

If $P(n) = a\,P(n-1) + b\,P(n-2)$ for $n \ge 2$; $P(0) = c$; $P(1) = d$ and m and N are defined by $B^2 = m + Ba + b$, $N = cm + dB + bc$, then

$$\frac{N}{Bm} = \sum_{i=1}^{\infty} \frac{P(i-1)}{B^i}$$ Long (1981)

provided that $|(a+\sqrt{(a^2+4b)})/(2B)| < 1$ *and* $|(a-\sqrt{(a^2+4b)})/(2B)| < 1$

8.3 Summations with fractions

$\sum_{i=0}^{\infty} \frac{F(i)}{2^i} = 2$	Vajda-60, Dunlap-51
$\sum_{i=0}^{\infty} \frac{L(i)}{2^i} = 6$	-
$\sum_{i=0}^{\infty} \frac{F(i)}{r^i} = \frac{r}{r^2 - r - 1}$	-
$\sum_{i=0}^{\infty} \frac{L(i)}{r^i} = 2 + \frac{r+2}{r^2 - r - 1}$	-
$\sum_{i=1}^{\infty} \frac{i\,F(i)}{2^i} = 10$	Vajda-61, Dunlap-52
$\sum_{i=1}^{\infty} \frac{i\,L(i)}{2^i} = 22$	-
$\sum_{i=\mathbf{0}}^{\infty} \frac{1}{F(\mathbf{2}^i)} = 4 - Phi = 3 - phi$	Vajda-77(corrected), Dunlap-53(corrected)
$\sum_{i=1}^{n} \frac{(-1)^{2^{i-1} r}}{F(2^i r)} = \frac{(-1)^r F(r(2^n-1))}{F(r)\,F(2^n r)}$	Vajda-89 (corrected)
$\sum_{k \ge 2} \frac{1}{F(k-1)F(k+1)} = 1$	R L Graham (1963) FQ 1.1, Problem B-9, pg 85, FQ 1.4 page 79
$\sum_{k \ge 2} \frac{F(k)}{F(k-1)F(k+1)} = 2$	R L Graham (1963) FQ 1.1, Problem B-9, pg 85
$\sum_{k \ge 2} \frac{(-1)^k}{F(k)F(k-1)} = phi$	Johnson-11, Vajda-102

8.4 Order 2 summations

$\sum_{i=1}^{n} F(i)^2 = F(n)\,F(n+1)$	Vajda-45, Dunlap-5, Hoggatt-I3, Lucas(1878), Koshy-77, B&Q(2003)-Identity 9 (Identity 233 variant)
$\sum_{i=1}^{n} L(i)^2 = L(n)\,L(n+1) - 2$	Hoggatt-I4

$$\sum_{i=0}^{2n-1} L(i)^2 = 5\,F(2n)\,F(2n-1)$$

-

$$\sum_{i=1}^{2n} F(i)\,F(i-1) = F(2n)^2$$

Vajda-40, Dunlap-45

$$\sum_{i=1}^{2n} L(i)\,L(i-1) = L(2n)^2 - 4$$

-

$$\sum_{i=1}^{2n+1} F(i)\,F(i-1) = F(2n+1)^2 - 1$$

Vajda-42, Dunlap-47

$$\sum_{i=1}^{2n+1} L(i)\,L(i-1) = L(2n+1)^2 - 1$$

-

$$5\sum_{k=0}^{n} (-1)^{r(1+k)}\,F(r(1+k))^2 = (-1)^{r(n+1)}\frac{F((2n+3)r)}{F(r)} - 2n - 3$$

Vajda-93

$$\sum_{k=0}^{n} (-1)^{r(1+k)}\,L(r(1+k))^2 = (-1)^{r(n+1)}\frac{F((2n+3)r)}{F(r)} + 2n + 1$$

Vajda-94

$$\sum_{i=0}^{n-1} F(2i+1)^2 = \frac{F(4n) + 2n}{5}$$

Vajda-95, B&Q(2003)-Identity 234

$$\sum_{i=0}^{n} F(2i)^2 = \frac{F(4n+2) - 2n - 1}{5}$$

Vajda page 70

$$\sum_{i=0}^{n-1} L(2i+1)^2 = F(4n) - 2n$$

Vajda-96, B&Q(2003)-Identity 54

$$\sum_{i=1}^{n} L(2i)^2 = F(4n+2) + 2n - 1$$

Vajda page 70

$$5\sum_{i=0}^{n} F(i)\,F(n-i) \begin{cases} = (n+1)\,L(n) - 2\,F(n+1) \\ = n\,L(n) - F(n) \end{cases}$$

Vajda-98, Dunlap-55, B&Q(2003)-Identity 58

$$\sum_{i=0}^{n} L(i)\,L(n-i) \begin{cases} = (n+1)\,L(n) + 2\,F(n+1) \\ = (n+2)\,L(n) + F(n) \end{cases}$$

Vajda-99, Dunlap-56, B&Q(2003)-Identity 57

$$\sum_{i=0}^{n} F(i)\,L(n-i) = (n+1)\,F(n)$$

Vajda-100, Dunlap-57, B&Q(2003)-Identity 35

$$\sum_{k=1}^{2n-1} (2n-k)\,F(k)^2 = F(2n)^2$$

V Hoggatt (1965) Problem B-53 FQ 3, pg 157

8.5 Summations of order > 2

$$10 \quad {}^{n} \quad F(i)^3 = F(3n+2) + 6(-1)^{n+1}F(n-1) + 5$$

adapted from Benjamin, Carnes, Cloitre (2009)

$$\sum_{i=1}$$

$$25\sum_{i=1}^{n} F(i)^4 = F(4n+2) + 4(-1)^{n+1}F(2n+1) + 6n + 3$$ see A005969

$$4\sum_{k=1}^{n} F(k)^6 = F(n)^5F(n+3) + F(2n)$$ Ohtsuka and Nakamura (2010) Theorem 1

$$4\sum_{k=1}^{n} L(k)^6 = L(n)^5L(n+3) + 125\,F(2n) - 128$$ Ohtsuka and Nakamura (2010) Theorem 2

8.6 G Summations

Two independent G series are denoted G(n) and H(n).

$\sum_{i=1}^{n} G(i) = G(n+2) - G(2)$	L G Brökling (1964) FQ 2.1 Problem B-20 solution, pg76; Vajda-33; Dunlap-38; B&Q(2003)-Identity 39
$\sum_{i=a}^{n} G(i) = G(n+2) - G(a+1)$	-
$\sum_{i=1}^{n} G(2i-1) = G(2n) - G(0)$	Vajda-34, Dunlap-37, B&Q(2003)-Identity 61
$\sum_{i=1}^{n} G(2i) = G(2n+1) - G(1)$	Vajda-35, Dunlap-39, B&Q(2003)-Identity 62
$\sum_{i=1}^{n} G(2i) - \sum_{i=1}^{n} G(2i-1) = \sum_{i=1}^{2n} (-1)^i G(i) = G(2n-1) + G(0) - G(1)$	Vajda-36, Dunlap-40
$\sum_{k=1}^{n} G(k-1)\,2^{-k} = (\,G(0) + G(3)\,)/2 - G(n+2)\,2^{-n}$	Vajda-37, Dunlap-41, B&Q(2003)-Identity 69
$\sum_{i=1}^{4n+2} G(i) = L(2n+1)\,G(2n+3)$	Vajda-38, Dunlap-43, B&Q(2003)-Identity 49
$\sum_{i=1}^{2n} G(i)\,G(i-1) = G(2n)^2 - G(0)^2$	Vajda-39, Dunlap-44, B&Q(2003)-Identity 41
$\sum_{i=1}^{2n+1} G(i)\,G(i-1) = G(2n+1)^2 - G(0)^2 - G(1)^2 + G(0)G(2)$	Vajda-41, Dunlap-46
$\sum_{i=1}^{n} G(i+2)\,G(i-1) = G(n+1)^2 - G(1)^2$	Vajda-43, Dunlap-48, B&Q(2003)-Identity 64
$(1 + (-1)^r - L(r))\sum_{k=0}^{n} G(m+kr) = G(m) - G(m+(n+1)r) + (-1)^r(G(m+nr) - G(m-r))$	**Fibonacci with a Golden Ring** Kung-Wei Yang *Mathematics Magazine* 70 (1997), pp. 131-135.

$$\sum_{i=1}^{n} G(i)^2 = G(n)\,G(n+1) - G(0)\,G(1)$$

Vajda-44, Dunlap-49, B&Q(2003)-Identity 67

$$\sum_{i=0}^{\infty} \frac{G(a,b,i)}{r^i} = a + \frac{a+b\,r}{r^2 - r - 1}$$

Stan Rabinowitz,
"Second-Order Linear Recurrences" card,
Generating Function
special case (x=1/r, P=1, Q=-1)

$$\sum_{i=0}^{\infty} \frac{i\,G(a,b,i)}{r^i} = \frac{r\,(b\,r^2 - 2\,a\,r + b - a)}{(r^2 - r - 1)^2}$$

-

$$\sum_{i=1}^{2n-1} G(i)\,H(i) = G(2n)\,H(2n-1) - G(0)\,H(1)$$

B&Q(2003)-Identity 42

8.7 Summations with Binomial Coefficients

$$\sum_{i=1}^{n} \binom{n-i}{i-1} = F(n)$$

B&Q(2003) Identity-4

$$\sum_{i=0}^{\infty} \binom{n-i-1}{i} = F(n)$$

Vajda-54(corrected),
Dunlap-84(corrected)

$$\sum_{i=0}^{n} \binom{n+i}{2i} = F(2n+1)$$

B&Q(2003)-Identity 165

$$\sum_{i=0}^{n-1} \binom{n+i}{2i+1} = F(2n)$$

B&Q(2003)-Identity 166

$$\sum_{k=0}^{n} \binom{n}{k} F(k) = F(2n)$$

S Basin & V Ivanoff (1963) Problem B-4, FQ 1.1 pg 74, FQ1.2 pg 79;
B&Q(2003)-Identity 6

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k+1} F(k) = F(n)$$

I Ruggles (1963) FQ 1.2 pg 77

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} L(k) = L(n)$$

I Ruggles (1963) FQ 1.2 pg 77

$$\sum_{k=0}^{n} \binom{n}{k} F(p-k) = F(p+n)$$

B&Q(2003)-Identity 20

$$\sum_{k=1}^{n} \binom{n}{k} 2^k F(k) = F(3n)$$

B&Q(2003)-Identity 238, Vajda-68

$$\sum_{i=0}^{n} \binom{n+1}{i+1} F(i) = F(2n+1) - 1$$

Vajda-50, Dunlap-82

$$\sum_{i=0}^{2n} \binom{2n}{i} F(2i+p) = 5^n F(2n+p)$$

Hoggatt-I41 (special case p=0: Vajda-69, Dunlap-85)

$$\sum_{i=0}^{2n} \binom{2n}{i} L(2i) = 5^n L(2n)$$

Vajda-71, Dunlap-87

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(2i+p) = 5^n L(2n+1+p)$$

Hoggatt-I42 (special case p=0: Vajda-70, Dunlap-86)

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(2i) = 5^{n+1} F(2n+1)$$

Vajda-72, Dunlap-88

$$\sum_{i=0}^{2n} \binom{2n}{i} F(i)^2 = 5^{n-1} L(2n)$$

Vajda-73, Dunlap-89,Hoggatt-I45

$$\sum_{i=0}^{2n} \binom{2n}{i} L(i)^2 = 5^n L(2n)$$

Vajda-75, Dunlap-91, Hoggatt-I46

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} F(i)^2 = 5^n F(2n+1)$$

Vajda-74, Dunlap-90, Hoggatt-I47

$$\sum_{i=0}^{2n+1} \binom{2n+1}{i} L(i)^2 = 5^{n+1} F(2n+1)$$

Vajda-76, Dunlap-92

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i+1} = 2^{n-1} F(n)$$

Vajda-91, B&Q(2003)-Identity 235, Catalan 1857

$$\sum_{i=0}^{\infty} 5^i \binom{n}{2i} = 2^{n-1} L(n)$$

Vajda-92, B&Q(2003)-Identity 237, Catalan (1857)-see Vajda pg 69

$$\sum_{i=0}^{k} \binom{k}{i} F(n)^i F(n-1)^{k-i} F(i) = F(kn)$$

Rabinowitz-17 (special case of Vajda-66)

$$\sum_{i=0}^{k} \binom{k}{i} F(n)^i F(n-1)^{k-i} L(i) = L(kn)$$

Rabinowitz-17 (special case of Vajda-66)

$$\sum_{i=0}^{p} \binom{p}{i} F(t)^i F(t-1)^{p-i} G(m+i) = G(m+tp)$$

Vajda-66

$$\sum_{i \ge 0} \sum_{j \ge 0} \binom{n-i}{j} \binom{n-j}{i} = F(2n+3)$$

B&Q(2003) Identity 5

$$\frac{F(rn)}{F(r)} = \sum_{k=0}^{[(n-1)/2]} (-1)^{k(r-1)} \binom{n-k-1}{k} L(r)^{n-1-2k}$$ Lucas (1878) equns 74-76, this form due to Hoggatt and Lindt (1969), see Gould (1977)

8.8 Powers of Fibonacci and Lucas as Sums

$$5^{k/2} F(t)^k = \sum_{i=0}^{(k-1)/2} \binom{k}{i} (-1)^{i(t+1)} \sqrt{5}\, F(\,(k-2i)t\,),\ k\ odd$$ Vajda-80

$$5^{k/2} F(t)^k = \sum_{i=0}^{k/2-1} \binom{k}{i} (-1)^{i(t+1)} L(\,(k-2i)t\,) + \binom{k}{k/2} (-1)^{(t+1)k/2},\ k\ even$$ Vajda-81

$$L(t)^k = \sum_{i=0}^{(k-1)/2} \binom{k}{i} (-1)^{it} L(\,(k-2i)t\,),\ k\ odd$$ Vajda-78

$$L(t)^k = \sum_{i=0}^{k/2-1} \binom{k}{i} (-1)^{it} L(\,(k-2i)t\,) + \binom{k}{k/2} (-1)^{tk/2},\ k\ even$$ Vajda-79

$$F_m^k F_n = (-1)^{kr} \sum_{h=0}^{k} \binom{k}{h} (-1)^h F_r^h F_{r+m}^{k-h} F_{n+kr+hm}$$ On a General Fibonacci Identity J H Halton, Fib Q, 3 (1965), pp 31-43

8.9 Summations with Binomials and G Series

$$\sum_{i=0}^{n} \binom{n}{i} G(i) = G(2n)$$ I Ruggles (1963) FQ 1.2 pg 77; Vajda-47; Dunlap-80

$$\sum_{i=0}^{n} \binom{n}{i} 2^i G(i) = G(3n)$$ B&Q(2003)-Identity 239

$$\sum_{i=0}^{n} \binom{n}{i} G(p-i) = G(p+n)$$ Vajda-46, Dunlap-79, B&Q(2003)-Identity 40

$$\sum_{i=0}^{n} \binom{n}{i} G(p+i) = G(p+2n)$$ Vajda-49, Dunlap-81

$$\sum_{i=0}^{p} (-1)^{p-i} \binom{p}{i} G(\,n+i\,) = G(\,n-p\,)$$ Vajda-51, Dunlap-83

9 Trigonometric Formulae

$$F(n) = \prod_{k=1}^{\lfloor (n-1)/2 \rfloor} 3 + 2\cos\frac{2k\pi}{n},\ n \ge 1$$ D Lind, Problem H-93, FQ 4 (1966), page 332

$$L(n) = \prod_{k=0}^{\lfloor (n-2)/2 \rfloor} 3 + 2\cos\frac{(2k+1)\pi}{n},\ n \ge 2$$ D Lind, Problem H-93, FQ 4 (1966), page 252, corrected page 332

10 E and Complex Numbers

$$\boldsymbol{i} = \sqrt{-1},\quad ln(x) = log_e(x)$$

Here we use g for ln(Phi), the natural log of Phi so that $\cosh(g) = \sqrt{5} / 2$.
Several of the formulae above can be derived using hyperbolic functions - see chapter XI of Vajda.

$$F(2n) = \frac{2}{\sqrt{5}} \sinh(2ng)$$ from Binet's formula

$$= \frac{\sinh(2ng)}{\cosh(g)}$$

$$F(2n+1) = \frac{2}{\sqrt{5}} \cosh((2n+1)g)$$ from Binet's formula

$$= \frac{\cosh((2n+1)g)}{\cosh(g)}$$

$L(2n) = 2\cosh(ng)$ from Binet's formula

$L(2n+1) = 2\sinh(ng)$ from Binet's formula

$\sin(\pi/2 + \boldsymbol{i}\, ln(\Phi)) = (\sqrt{5})/2 = \Phi + ½$ Schroeder 1986, equation (5.41) page 68

$$\sum_{n=0}^{\infty} \frac{F(n)}{n!} = \frac{e^{\Phi} - e^{-\phi}}{\sqrt{5}}$$ a special case of the EGF - see next page
C. Brown Jan 2015 private communication

$$\sum_{n=0}^{\infty} \frac{L(n)}{n!} = e^{\Phi} + e^{-\phi}$$ a special case of the EGF - see next page
C. Brown Jan 2015 private communication

$$\sum_{i=1}^{\infty} \frac{\Phi F(i) - F(i+1)}{i} = \sum_{i=1}^{\infty} \frac{\sqrt{5} F(i) - L(i)}{2i} = g$$ C. Brown (Jan 2016) private communication

$$F(n) = \prod_{k=1}^{n-1} \left(1 - 2\,\boldsymbol{i} \cos\frac{k\pi}{n}\right)$$ D Lind, Problem H-64, FQ 3 (1965), page 116

$$F(n) = \frac{2\,\boldsymbol{i}^{1-n}}{\sqrt{5}} \sin(-\boldsymbol{i}\, n\, ln(\boldsymbol{i}\, Phi))$$ from Rabinowitz-7 corrected, using $Phi^2 = (\sqrt{5}+1)/(\sqrt{5}-1)$

$$F(n) = \frac{2\,\boldsymbol{i}^{-n}}{\sqrt{5}} \sinh(n\, ln(\boldsymbol{i}\, Phi))$$ from Rabinowitz-7 corrected

$L(n) = 2\,\boldsymbol{i}^{-n} \cos(-\boldsymbol{i}\, n\, ln(\boldsymbol{i}\, Phi))$ from Rabinowitz-7 corrected

$L(n) = 2\,\boldsymbol{i}^{-n} \cosh(n\, ln(\boldsymbol{i}\, Phi))$ from Rabinowitz-7 corrected

$\sqrt{1 + 2\boldsymbol{i}} = \sqrt{Phi} + \boldsymbol{i}\sqrt{phi}$
I J Good (1993)

$= [\,1+\boldsymbol{i};\ \overline{2 + 2\boldsymbol{i}}\,]$

$\sqrt{1 + \boldsymbol{i}/2} = \left(\sqrt{\sqrt{5}+2} + \boldsymbol{i}\sqrt{\sqrt{5}-2}\,\right)/2$

$= (Phi^{3/2} + \boldsymbol{i}\, phi^{3/2})/2$ I J Good (1993)

$= [\,1/2 + \boldsymbol{i}/2;\ \overline{1+\boldsymbol{i}}\,]$

11 Generating Functions

This section is now part of the following reference page on Linear Recurrence Relations and Generating Functions

12 References

* in references above indicates a private communication.
: a book

: an article in a journal

: a link to a web resource

FQ : The Fibonacci Quarterly journal
All papers in the last 7 years are only available online to subscribers but older ones have free access in PDF form (shown with links below).

Arranged in alphabetical order of author:

A T Benjamin, J J Quinn Proofs That Really Count Mathematical Association of America, 2003, ISBN 0-88385-333-7, hardback, 194 pages. shown as B&Q(2003) in the Table above
Art Benjamin and Jennifer Quinn have a wonderful knack of presenting proofs that involve counting arrangements of dominoes and tiling patterns in two ways that convince you that a formula really *is* true and not just "proved"! The identities proved mainly involve Fibonacci, Lucas and the General Fibonacci series with chapters on continued fractions, binomial identites, the Harmonic and Stirling numbers too. There is so much here to inspire students to find proofs for themselves and to show that proofs can be fun too!
Important notation difference: Benjamin and Quinn use f_n for the Fibonacci number F(n+1)

Bergum and Hoggatt (1975)
G. E. Bergum and V. E. Hoggatt, Jr. "Sums and Products for Recurring Sequences" *Fib Q* 13 (1975), pages 115-120 free pdf

Benjamin, Carnes, Cloitre (2009)
"Recounting the Sums of Cubes of Fibonacci Numbers" A T Benjamin, T A. Carnes, B Cloitre, *Congressus Numerantium, Proceedings of the Eleventh International Conference on Fibonacci Numbers and their Applications*, William Webb (ed.), Vol 194, pp. 45-51, 2009.

Binet (1843) J P M Binet, page 563 of *Comptes Rendus*, Vol 17, Paris, 1843

Bro A Brousseau (1968) A Sequence of Power Formulas *The Fibonacci Quarterly* vol 6 pages 81-83 as pdf
gives the recurrence relations of powers of Fibonacci's in terms of Fibonomials, as developed at the start of this page, but without explicitly stating the general formula and without recognizing the Fibonomials.

L E Dickson History of the Theory of Numbers: Vol 1 Divisibility and Primality
is a classic and monumental reference work on all aspects of Number Theory in 3 volumes (volume II is on Diophantine Analysis and volume III on Quadratic and Higher Forms). Although not up-to-date (the original edition was 1952) it is still a comprehensive summary of useful historical and early references on all aspects of Number Theory. The link is to a new cheap Dover paperback edition (2005) of Volume 1 which contains the most about Fibonacci Numbers, Lucas numbers and the golden section: see Chapter XV11 on **Recurring Series, Lucas' u_n, v_n** where he uses *the series of Pisano* for what we now call the *Fibonacci numbers*.

R A Dunlap, The Golden Ratio and Fibonacci Numbers World Scientific Press, 1997, 162 pages.
An introductory book strong on the geometry and natural aspects of the golden section but it does not include much on the mathematical detail. Beware - some of the formulae in the Appendix are wrong! Dunlap has copied them from Vajda's book (see below) and he has faithfully preserved all of the original errors! The formulae on this page (that you are now reading) are corrected versions and have been verified.

Fairgrieve and Gould (2005)

"Product Difference Fibonacci Identities of Simson, Gelin-Cesáro, Tagiuri and Generalizations" S Fairgrieve and H W Gould, *The Fibonacci Quarterly* vol 43 (2005), 137-141. free pdf

Gould (1977) "A Fibonacci Formula of Lucas and its Subsequent Manifestations and Rediscoveries" H W Gould, *Fibonacci Quarterly* vol 15 (1977) pages 25-29 free pdf

R L Graham, D E Knuth, O Patashnik Concrete Mathematics Second Edition (1994), hardback, Addison-Wesley.
No - this is not a book about proportions of sand to cement 🙂. The title is meant as an antidote to the "Abstract Mathematics" courses so often found in the curricullum of a university maths degree. As such, it is ***the*** book to dip into if you want to go really deeply into any part of the mathematics covered on this Fibonacci and Phi site. However, it quickly gets to an advanced mathematics undergraduate level after some nice introductions to every topic.
There are notes left in the margins which were inserted by students taking the original courses based on this book at Stanford university and they are interesting, often useful and sometimes quite funny.

Hansen (1972)
"Generating Identities for Fibonacci and Lucas Triples" Rodney T Hansen, FQ (1972), pages 571-578 pdf

V E Hoggatt Jr "Fibonacci and Lucas Numbers" published by The Fibonacci Association, 1969 (Houghton Mifflin) (free online)
A very good introduction to the Fibonacci and Lucas Numbers written by a founder of the Fibonacci Quarterly.

Hoggatt and Lind (1969) "Compositions and Fibonacci Numbers" V E Hoggatt Jr, D A Lind, *The Fibonacci Quarterly*, Vol. 7, No. 3 (Oct., 1969), pp. 253-266. free pdf

F T Howard (2003) "The Sum of the Squares of Two Generalized Fibonacci Numbers" *FQ* vol 41 pages 80-84, pdf

Horadam (1967)
A F Horadam "Special Properties of the Sequence $w_n(a,b;p,q)$" *FQ* 5 (1967) pgs 424-434 pdf

Hudson and Winans (1981)
"A Complete Characterization of the Decimal Fractions That Can Be Represented as $\Sigma\ 10^{k(a+1)}F_{ai}$, where F_{ai} is the ai^{th} Fibonacci Number" R H Hudson, C F Winans *The Fibonacci Quarterly* 19, no. 5 (1981) pages 414-421. free pdf
See also
A Complete Characterization Of B-Power Fractions That Can Be Represented As Series Of General N-Bonacci Numbers J-Z Lee, J-S Lee *Fibonacci Quarterly* 25 (1987) pages 72-75. free pdf

I J Good "Complex Fibonacci And Lucas Numbers, Continued Fractions and The Square Root Of The Golden Ratio", *Fib Q* 31 (1993) pages 7-19 free pdf and corrections

R Johnson (Durham university) has an excellent web page
on the power of matrix methods to establish many Fibonacci formula with ease (but it does rely on at least undergraduate level matrix mathematics). See the **Matrix methods for Fibonacci and Related Sequences** link to a Postscript and PDF version on his Fibonacci Resources web page.
The latest version (Nov 12, 2004) contains an appendix showing how formulae developed in Johnson's paper can prove almost all the identities here in my table above.

D E Knuth The Art of Computer Programming: Vol 1 Fundamental Algorithms hardback, Addison-Wesley third edition (1997).
The paperback is now out of print but the title link is to hardcover, second hand and Kindle versions. This is the first of three volumes and an absolute must for all computer scientist/mathematicians.

T Koshy Fibonacci and Lucas Numbers with Applications, Wiley-Interscience, 2001, 648 pages.
This book is packed full of an amazing number of Fibonacci and related equations, mostly culled from the pages of the *Fibonacci Quarterly*. Although initially impressive in its size and breadth, be aware that there are far too many typos, errors and missing or irrelevant conditions in many of its formulae as well as some glaring omissions and misattributions particularly with respect to the original references for a number of the formulae. Although Fibonacci representations of integers are included Zeckendorf himself is never even mentioned! There are lots of exercises with answers to the odd-numbered questions.

Long (1981) The Decimal Expansion Of 1/89 And Related Results, C Long *Fibonacci Quarterly* vol 19 (1985) pages 53-55 free pdf

Long (1986)
Discovering Fibonacci Identities, FQ 24 (1986), pages 160-166 pdf

Lucas (1876)
E Lucas, in *Nuov. Corresp. Math.* 2 (1876) , pages 74-75
See Dickson Vol 1 page 395

E Lucas, "Théorie des Fonctions Numériques Simplement Périodiques" in *American Journal of Mathematics* vol 1 (1878) pages 184-240 and 289-321.
Reprinted in English translation as The Theory of Simply Periodic Functions, the Fibonacci Association, 1969 free pdf

R S Melham (1999) "Families of Identities Involving Sums of Powers of the Fibonacci and Lucas Numbers" *FQ* vol 37, pages 315-319 free pdf

R S Melham (2011), On Product Difference Fibonacci Identities Article A10, Integers, vol 11

Morgado (1987)
J Morgado "Note on some results of A F Horadam and A G Shannon concerning Catalan's Identity on Fibonacci Numbers" *Portugaliae Math.* 44 (1987) pgs 243-252 pdf

Ohtsuka and Nakamura (2010)
"A New Formula For The Sum Of The Sixth Powers Of Fibonacci Numbers" H Ohtsuka, S Nakamura, *Congressus Numerantium* Vol. 201 (2010), Proceedings of the Thirteenth Conference on Fibonacci Numbers and their Applications , pp.297-300.

S Rabinowitz "Algorithmic Manipulation of Fibonacci Identities" in Applications of Fibonacci Numbers: Proceedings of the Sixth International Research Conference on Fibonacci Numbers and their Applications, editors G E Bergum, A N Philippou, A F Horodam; Kluwer Academic (1996), pages 389 - 408.

Schroeder 1986
Number Theory in Science and Communication, With Applications in Cryptography, M R Schroeder, Springer-Verlag (2nd enlarged edition) 1986.

B Sharpe (1965)
On Sums $F_x^2 \pm F_y^2$ *Fib Quart* (1965) 3.1 page 63 pdf

S Vajda, Fibonacci and Lucas numbers, and the Golden Section: Theory and Applications, Dover Press (2008).
This is a wonderful book, a classic, originally published in 1989 and now back in print in this Dover edition. This book is full of formulae on the Fibonacci numbers and Phi and the Lucas numbers. The whole book develops the formulae step by step, proving each from earlier ones or occasionally from scratch. It has a few errors in its formulae and all of them have been dutifully and exactly copied by authors such as Dunlap (see above) and others! Where I have identified errors, ***they have been corrected here on this page***.

Fibonacci Home Page 1.61803

This is the first page on this Topic.

Fibonacci and Phi in the Arts

WHERE TO NOW???

More topics at Ron Knott's Home page

Recurrence Relations & Generating Functions