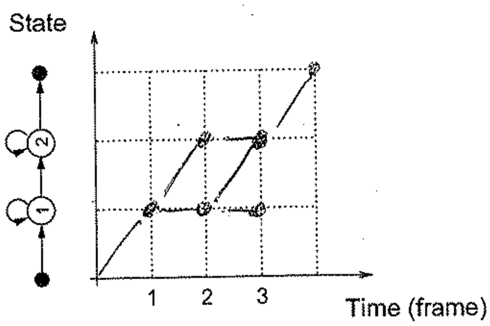
Worked example of the forward procedure

$$\pi = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0.8 & 0.2 \\ 0 & 0.6 \end{bmatrix} \quad \eta = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}$$



$$\begin{array}{cc} & \begin{array}{cccc} R & G & B & Y \end{array} \\ B = & \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0 \\ 0 & 0.8 & 0.1 & 0.1 \end{bmatrix} \end{array}$$

$$O = \{R, G, G\}$$

$t=1$:

$$\alpha_1(1) = 1 \times 0.5 = 0.5$$
$$\alpha_1(2) = 0 \times 0 = 0$$

$t=2$:

$$\alpha_2(1) = [\alpha_1(1) \times 0.8] \times 0.2 = 0.08$$
$$\alpha_2(2) = [\alpha_1(1) \times 0.2] \times 0.8 = 0.08$$

$t=3$:

$$\alpha_3(1) = [\alpha_2(1) \times 0.8] \times 0.2 = 0.0128$$
$$\alpha_3(2) = [\alpha_2(1) \times 0.2 + \alpha_2(2) \times 0.6] \times 0.8 = 0.0512$$

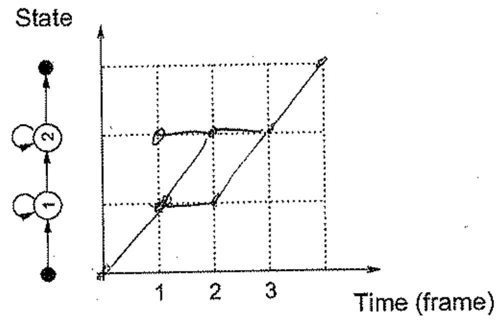
$$P(O|\lambda) = \alpha_3(2) \times 0.4 = 0.02048$$

Worked example of the backward procedure

$$\pi = [1 \quad 0]$$

$$A = \begin{bmatrix} 0.8 & 0.2 \\ 0 & 0.6 \end{bmatrix} \qquad \eta = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}$$

$$B = \begin{array}{cccc} R & G & B & Y \\ \multicolumn{4}{c}{\begin{bmatrix} 0.5 & 0.2 & 0.3 & 0 \\ 0 & 0.8 & 0.1 & 0.1 \end{bmatrix}} \end{array}$$



$$O = \{R, G, G\}$$

$$\beta_3(1) = 0 \qquad \beta_2(1) = 0.2 \times 0.8 \times 0.4 = 0.064$$

$$\beta_3(2) = 0.4 \qquad \beta_2(2) = 0.6 \times 0.8 \times 0.4 = 0.192$$

$$\beta_1(1) = [a_{11}\, b_1(G)\, \beta_2(1) + a_{12}\, b_2(G)\, \beta_2(2)]$$
$$= 0.8 \times 0.2 \times 0.064 + 0.2 \times 0.8 \times 0.192$$
$$= 0.01024 + 0.03072$$
$$= 0.04096$$

$$\beta_1(2) = [a_{22}\, b_2(G)\, \beta_2(2)]$$
$$= 0.6 \times 0.8 \times 0.192 = 0.09216$$

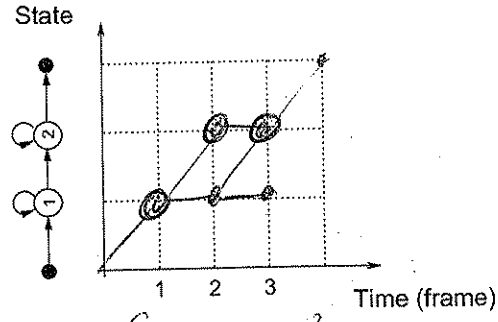
$$P(O|\lambda) = [\pi_1\, b_1(R)\, \beta_1(1) + \pi_2\, \cancel{b_2(R)\, \beta_1(2)}]$$
$$= 1 \times 0.5 \times 0.04096$$
$$= 0.02048$$

Worked example of Viterbi algorithm

$\pi = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0.8 & 0.2 \\ 0 & 0.6 \end{bmatrix} \quad \eta = \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}$$

$$B = \begin{array}{c} \begin{bmatrix} 0.5 & 0.2 & 0.3 & 0 \\ 0 & 0.8 & 0.1 & 0.1 \end{bmatrix} \\ \begin{matrix} R & \;\;G & \;\;B & \;\;Y \end{matrix} \end{array}$$



$O = \{R, G, G\}$

$t=1$:

$$\delta_1(1) = 1 \times 0.5 = 0.5$$
$$\delta_1(2) = 0 \times 0 = 0$$

$t=2$:

$$\delta_2(1) = [\delta_1(1) \times 0.8] \times 0.2 = 0.08$$
$$\delta_2(2) = [\delta_1(1) \times 0.2] \times 0.8 = 0.08$$

$t=3$:

$$\delta_3(1) = [\delta_2(1) \times 0.8] \times 0.2 = 0.0128$$
$$\delta_3(2) = \max[\delta_2(1) \times 0.2,\ \underline{\delta_2(2) \times 0.6}] \times 0.8 = 0.0384$$

Finally,

$$\Delta^*_T = P(O, X^* \mid \lambda) = \delta_3(2) \times 0.4 = 0.01536$$

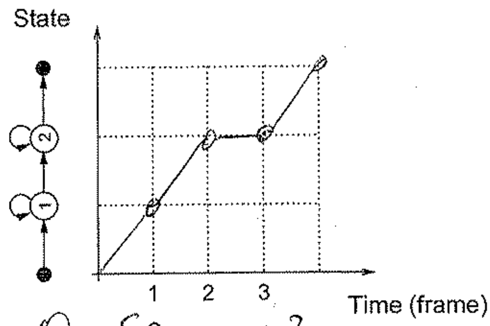
Thus, the best path $X^* = \{1, 2, 2\}$, which happens to match the actual state sequence that we used to generate the observations in the first place.

Worked example of Viterbi re-estimation

$q_1(1) = 1 \qquad q_1(2) = 0$

$q_2(1) = 0 \qquad q_2(2) = 1$

$q_3(1) = 0 \qquad q_3(2) = 1$



$O = \{R, G, G\}$

$$\hat{\pi} = [1 \quad 0]$$

$$\hat{A} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\hat{B} = \begin{array}{c} \\ \\ \end{array}\begin{array}{cccc} R & G & B & Y \end{array} \\ \hat{B} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Worked example of Baum-Welch re-estimation

$\alpha_1(1) = 0.5$
$\alpha_1(2) = 0$

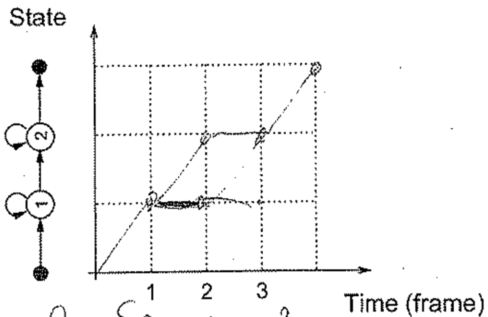
$\alpha_2(1) = 0.08$
$\alpha_2(2) = 0.08$

$\alpha_3(1) = 0.0128$
$\alpha_3(2) = 0.0512$

$\beta_1(1) = 0.04096$
$\beta_1(2) = 0.09216$

$\beta_2(1) = 0.064$
$\beta_2(2) = 0.192$

$\beta_3(1) = 0$
$\beta_3(2) = 0.4$



$O = \{R, G, G\}$

$$\gamma_t(i) = \frac{\alpha_t(i)\,\beta_t(i)}{P(O|\lambda)}$$

$$\xi_t(i,j) = \frac{\alpha_{t-1}(i)\, a_{ij}\, b_j(o_t)\,\beta_t(j)}{P(O|\lambda)}$$

$P(O|\lambda) = 0.02048$

$$\gamma_1(1) = \frac{0.5 \times 0.04096}{0.02048} = 1$$

$$\gamma_1(2) = \frac{0 \times 0.09216}{0.02048} = 0$$

$$\gamma_2(1) = \frac{0.08 \times 0.064}{0.02048} = 0.25$$

$$\gamma_2(2) = \frac{0.08 \times 0.192}{0.02048} = 0.75$$

$$\gamma_3(1) = \frac{0.0128 \times 0}{0.02048} = 0$$

$$\gamma_3(2) = \frac{0.0512 \times 0.4}{0.02048} = 1$$

Considering only the non-zero prob. transitions:

$$\xi_2(1,1) = \frac{0.5 \times 0.8 \times 0.2 \times 0.064}{0.02048} = 0.25$$

$$\xi_2(1,2) = \frac{0.5 \times 0.2 \times 0.8 \times 0.192}{0.02048} = 0.75$$

$$\xi_3(1,2) = \frac{0.08 \times 0.2 \times 0.8 \times 0.4}{0.02048} = 0.25$$

$$\xi_3(2,2) = \frac{0.08 \times 0.6 \times 0.8 \times 0.4}{0.02048} = 0.75$$

$$\hat{\pi} = [1 \quad 0]$$

$$\hat{B} = \begin{array}{cccc} R & G & B & Y \end{array}$$
$$\hat{B} = \begin{bmatrix} 0.8 & 0.2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\hat{A} = \begin{bmatrix} \frac{0.25}{1.25} & \frac{0.75 + 0.25}{1.25} \\ 0 & \frac{0.75}{0.75} \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0 & 1 \end{bmatrix}$$