1. $z = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \\ -1 \end{bmatrix}$ $y = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

(i) $z+y = \begin{bmatrix} 3 \\ 3 \\ -2 \\ 2 \\ -2 \end{bmatrix}$ (ii) $z-y = \begin{bmatrix} -1 \\ -3 \\ -4 \\ 2 \\ 0 \end{bmatrix}$ (iii) $z^T y = \begin{bmatrix} 1 & 0 & -3 & 2 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \\ -1 \end{bmatrix}$

$= 2+0+(-3)+0+1$

$= 0$

(iv) $zy^T = \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \\ -1 \end{bmatrix} \begin{bmatrix} 2 & 3 & 1 & 0 & -1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ -6 & -9 & -3 & 0 & 3 \\ 4 & 6 & 2 & 0 & -2 \\ -2 & -3 & 1 & 0 & 1 \end{bmatrix}$

2. $\Sigma = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix}$

(i) $\Sigma^T = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix}$ (Σ is symmetric, so $\Sigma = \Sigma'$)

(ii) $\det \Sigma = |\Sigma|$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

$$|\Sigma| = \begin{vmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{vmatrix}$$

$$= 4\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} - \begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix}$$

$$= 4(6-1) - (3-3) - 3(-1+6)$$

$$= 20 - 15$$

$$= 5.$$

(iii) $|\Sigma| = |\Sigma^T| = 5$

(iv) $Tr(\Sigma) = 4+2+3$
$= 9$

$$\boxed{Tr(A) = \sum_{i=1}^{n} a_{ii}}$$

(v) $\Sigma^{-1} = \dfrac{1}{|\Sigma|}(C_{ij})^T$ $\quad$ C_{ij} is a cofactor

$$= \frac{1}{|\Sigma|} \underbrace{\begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}}_{\text{adjugate matrix}}$$

$$= \frac{1}{5}\begin{bmatrix} +\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ -1 & 3 \end{vmatrix} & +\begin{vmatrix} 1 & -3 \\ 2 & -1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ -3 & 3 \end{vmatrix} & +\begin{vmatrix} 4 & -3 \\ -3 & 3 \end{vmatrix} & -\begin{vmatrix} 4 & -3 \\ 1 & -1 \end{vmatrix} \\ +\begin{vmatrix} 1 & 2 \\ -3 & -1 \end{vmatrix} & -\begin{vmatrix} 4 & 1 \\ -3 & -1 \end{vmatrix} & +\begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 & 5 \\ 0 & 3 & 1 \\ 5 & 1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3/5 & 1/5 \\ 1 & 1/5 & 7/5 \end{bmatrix}$$

(vi) $\Sigma \times \Sigma = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix}$

$= \begin{bmatrix} 26 & 9 & -22 \\ 9 & 6 & -8 \\ -22 & -8 & 19 \end{bmatrix}$

(vii) $\det(3\Sigma) = 3^3 |\Sigma|$

$= 27 \times 5 = 135$

$\boxed{|aA| = a^n |A|}$

3. $x = [1 \ -1 \ 2]^T$

(i) $\Sigma x = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$= \begin{bmatrix} -3 \\ -3 \\ 4 \end{bmatrix}$

(ii) $x^T \Sigma x = [1 \ -1 \ 2] \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$= [1 \ -1 \ 2] \begin{bmatrix} -3 \\ -3 \\ 4 \end{bmatrix}$

$= 8$

(iii) $x^T \Sigma^{-1} x = [1 \ -1 \ 2] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3/5 & 1/5 \\ 1 & 1/5 & 7/5 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$

$= [1 \ -1 \ 2] \begin{bmatrix} 3 \\ -1/5 \\ 3\,3/5 \end{bmatrix}$

$= 10\,2/5.$

(iv) $\det\{xx^T \Sigma\} = |xx^T|\,|\Sigma|$

$|A||B| = |A \cdot B|$
(distributive)

$$= \begin{vmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{vmatrix} \cdot 5$$

$$= \left\{ + \begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} + \begin{vmatrix} -1 & -2 \\ 2 & 4 \end{vmatrix} + 2 \begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix} \right\} \times 5$$

$$= 0 \times 5 = 0.$$

(v) $\mathrm{Tr}\{xx^T \Sigma\} = \mathrm{Tr}\left\{ \begin{bmatrix} 1 & -1 & 2 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{bmatrix} \begin{bmatrix} 4 & 1 & 3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix} \right\}$

$$= \mathrm{Tr} \left\{ \begin{matrix} -3 & -3 & 4 \\ 3 & 3 & -4 \\ -6 & -6 & 8 \end{matrix} \right\} = 8.$$

4. $\Sigma_1 = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix}$ $\quad \Sigma_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

(i) $\det \Sigma_2 = 2 \begin{vmatrix} 3 & 0 \\ 0 & 3 \end{vmatrix} + \begin{vmatrix} -1 & 0 \\ 0 & 3 \end{vmatrix} + 0$

$= 18 - 3 = 15$

(ii) $\Sigma_2^{-1} = \dfrac{1}{|\Sigma_2|} \begin{vmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{vmatrix}$

$= \dfrac{1}{15} \begin{bmatrix} 9 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

(iii) $Tr(\Sigma_2) = 2 + 3 + 3 = 8$.

(iv) Show that $(\Sigma_1 \Sigma_2)^{-1} = \Sigma_2^{-1} \Sigma_1^{-1}$

$(\Sigma_1 \Sigma_2)^{-1} = \begin{bmatrix} 7 & -1 & -9 \\ 0 & 5 & -3 \\ -5 & 0 & 9 \end{bmatrix}^{-1} = \dfrac{adj(A)}{\det(A)}$

$\det A_* = 75$

$adj(A_*) = \begin{bmatrix} 45 & 9 & 48 \\ 15 & 18 & 21 \\ 25 & 5 & 35 \end{bmatrix}$

$\Sigma_2^{-1} = \dfrac{adj(A_2)}{\det(A_2)} = \dfrac{1}{15} \begin{bmatrix} 9 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$\Sigma_2^{-1} \Sigma_1^{-1} = \dfrac{1}{\det(A_1)} \cdot \dfrac{1}{\det(A_2)} \, adj(A_2) \, adj(A_1) = \dfrac{1}{5} \times \dfrac{1}{15} \begin{bmatrix} 9 & 3 & 0 \\ 3 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 0 & 5 \\ 0 & 3 & 1 \\ 5 & 1 & 7 \end{bmatrix}$

$= \dfrac{1}{75} \begin{bmatrix} 45 & 9 & 48 \\ 15 & 18 & 21 \\ 25 & 5 & 35 \end{bmatrix}$

4 (v) $\Sigma_1 \Sigma_1^{-1} = I$

$$\begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & -1 \\ -3 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3/5 & 1/5 \\ 1 & 1/5 & 7/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4 (vi) Σ_2 is positive definite.

① $x^T \Sigma_2 x > 0$. for any x.

② All eigenvalues λ_i of Σ_2 are positive.

4(vi). To show $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is positive definite,

~~it~~ will satisfy $\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \geq 0 \longrightarrow (1)$

$$\begin{bmatrix} 2a-b & -a+3b & 3c \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} \geq 0$$

$$2a^2 - ab - ab + 3b^2 + 3c^2 \geq 0 \qquad \cancel{\times \tfrac{1}{2}}$$

$$2a^2 - 2ab + 3b^2 + 3c^2 \geq 0 \qquad \times \tfrac{1}{2}$$

$$a^2 - ab + \tfrac{3}{2}b^2 + \tfrac{3}{2}c^2 \geq 0. \longrightarrow (2)$$

Note that $(a - \frac{b}{2})^2 = a^2 - ab + \frac{b^2}{4}$.

So (2) can be rewritten as:

$$(a - \tfrac{b}{2})^2 + (\tfrac{3}{2} - \tfrac{1}{4})b^2 + \tfrac{3}{2}c^2 \geq 0.$$

$$\underbrace{(a - \tfrac{b}{2})^2}_{\text{positive}} + \underbrace{\tfrac{5}{4}b^2}_{\text{positive}} + \underbrace{\tfrac{3}{2}c^2}_{\text{positive.}} \geq 0.$$

$$\Sigma = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$$

$$p(\lambda) = |\Sigma - \lambda I|$$
$$= \left| \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$
$$= \begin{vmatrix} 1.5-\lambda & 0.5 \\ 0.5 & 1.5-\lambda \end{vmatrix}$$
$$= (1.5-\lambda)^2 - (0.5)^2$$

$$p(\lambda) = 0 \Rightarrow \left(\tfrac{3}{2}-\lambda\right)^2 = \tfrac{1}{4}$$
$$\tfrac{3}{2}-\lambda = \pm\tfrac{1}{2}$$
$$\therefore \lambda = \{1, 2\}.$$

$$\boxed{\Sigma v = \lambda v \quad \text{or} \quad \begin{aligned} \Sigma v - \lambda v &= 0 \\ (\Sigma - \lambda I)v &= 0 \end{aligned}}$$

When $\lambda = 1$, $(\Sigma - \lambda I)v = 0$

$$\begin{bmatrix} 1.5-\lambda & 0.5 \\ 0.5 & 1.5-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow \tfrac{1}{2}v_1 + \tfrac{1}{2}v_2 = 0 \Rightarrow V_1 = -V_2$$

Hence the associated eigen vector is $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

(we parameterise v_1 according to v_2
Say, if $v_2 = t$, then $v_1 = -t$)

When $\lambda = 2$, from: $(\Sigma - \lambda I)v = 0$

$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

$$\Rightarrow -\frac{1}{2}v_1 + \frac{1}{2}v_2 = 0$$

so $v_1 = v_2$ and the associated eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$\therefore$ The matrix $\begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix}$ has $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ eigenvectors and $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ eigen values.

The eigenvectors $\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ is not normalised. We can normalise them by $\begin{bmatrix} \frac{u_1}{\|u_1\|} & \frac{u_2}{\|u_2\|} \end{bmatrix}$

where $u_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

The normalised eigenvectors are $\frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$ since

$$\|u_1\| = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$$
$$\|u_2\| = \sqrt{(1)^2 + (1)^2} = \sqrt{2}.$$

7. $w = U^T \quad x$ $\qquad\qquad\qquad\qquad$ $\left(\frac{2}{\sqrt{2}}\right)^2 = \frac{4}{2} = 2.$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$= \frac{1}{\sqrt{2}}\begin{bmatrix} -2 \\ 0 \end{bmatrix}$ $\qquad\qquad$ $\|w\| = \sqrt{\frac{2^2}{\sqrt{2}} + 0} = \sqrt{2}.$

$\|x\| = \frac{1}{\sqrt{(1^2)+(-1)^2}} = \frac{1}{\sqrt{2}}$

$\|w\| = \|x\|$

8. $\|z-y\| = \sqrt{(1-2)^2 + (0-3)^2 + (-3-1)^2 + (2-0)^2 + (-1+1)^2}$

$= \sqrt{1+9+16+4+0}$

$= \sqrt{30}$