

Small antenna input impedances that contradict Chu-Wheeler Q criterion

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Analysis of the small tuned folded dipole input impedance gives a limiting bandwidth proportional to frequency and antenna dimension. A simple antenna mode expansion also directly contradicts the Chu-Wheeler Q criterion. A line-integral potential method gives an additional logarithmic Q reduction factor. Untuned field sensors may have additional coupling losses.

Introduction: The Chu-Wheeler criterion states that the lowest obtainable Q factor for any antenna is $Q = (ka)^{-3}$ where k is the propagation constant $2\pi/\lambda$, λ is the (free space) wavelength, and a is the radius of the sphere just containing the antenna. Chu [1] obtained this result by assuming the stored energy and radiation surrounding the sphere to be antenna modes with integer order spherical Bessel (Hankel) function expansions. He assumed an equivalent circuit based on a partial fraction expansion of the most significant modes and computed the limiting Q from this. Wheeler [2] considered radiating capacitors and inductors with shapes and forms that filled the sphere as far as possible optimally. His results agreed well with Chu's assertion. More recently MacLean [3] has proposed a modification that eased the criterion for ka values around $ka = 1$. $ka \leq 1$ is normally considered to be the limit of a 'small' antenna. Subsequent to the original proposal Hansen [4] implied, and others [5] stated more categorically, that all small antennas obeyed the Chu-Wheeler criterion without exception. This is challenged by the results of Grimes and Grimes [6], mainly in the case of multi-element antennas, and their criteria tend towards $Q = (ka)^{-1}$ in some regions. Grimes and Grimes do however support Chu's results as originally stated.

The five various examples given here show that the Chu-Wheeler criterion is usually very significantly over-shadowed by less stringent Q criteria depending on the antenna type and size. However the $Q = (ka)^{-1}$ criterion is seen to be generally slightly more 'benign' and less strict than those derived here. The selected examples agree within a maximum error of about 20% with our previously reported loop measurements [7].

We also suggest that the coupling factors and hence computed sensitivities of untuned or active small antennas, used as field sensors, are too low by up to about $1/2\pi$ or 16 dB unless they have been calibrated against resonant antennas of known sensitivity and efficiency.

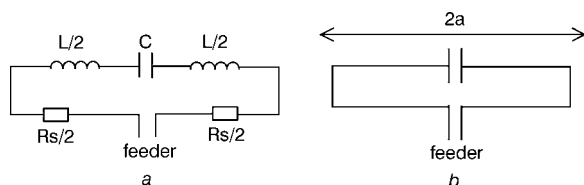


Fig. 1 *Folded dipole*
 a Equivalent circuit
 b Small tuned folded dipole

Example 1—Tuned folded dipole: Fig. 1 shows the equivalent circuit of the small tuned folded dipole. For convenience, the dipole is taken to have conductor thickness and spacing such that it can be analysed as two pieces of 300 Ω twin transmission line. With sufficient conductor coupling, the dipole mode currents are equal and in phase in the lower and upper conductors so that a four-to-one impedance step-up applies to the assumed theoretical small dipole mode radiation resistance. Then for small ka :

$$X_L = 2 \times 300 \tan(ka) \cong 600 ka \quad (1)$$

$$R_{rad} \cong 4 \times 20 (ka)^2 = 80 (ka)^2 \quad (2)$$

$$Q_1 = \frac{X_L}{R_{rad}} = \frac{7.5}{ka} = \frac{1.2\lambda}{a} \quad (3)$$

The Q factor for this model tends towards $7.5/ka = 1.2\lambda/a$. Thus for this 'practical' semi-empirical model, based on no new theory, the Chu-Wheeler limit does not apply for antenna sizes where $ka < 1/\sqrt{7.5}$. This result is modified to be usefully less stringent by the 'logarithmic' proposal below.

Examples 2 and 3—(External) antenna mode potentials: As in Chu [1] we assume that antenna modes exist surrounding the sphere containing the antenna. A uniform current density is assumed to cover the surface of the sphere, creating the field and potential distributions outside the sphere. Importantly all current elements are assumed to be parallel, so that at the poles of the sphere (assumed to be vertical) the current elements are directed outwards and correspond to power flowing outwards as displacement currents. The currents at the equator radiate horizontally as is the traditional convention. The field and potential distributions outside the sphere are assumed to be direct continuations of the fields and potentials of equivalent point sources at the centre of the sphere.

We consider two different cases. In the first case (a), we assume that the potential travels with the velocity of light c_{em} from the central source to the surface of the sphere and the central source is in phase with currents on the surface. In the second case (b), we assume that the field travels from the source with the velocity of light and obeys the inverse square distance law, and that we can then obtain the potential by integrating the field radially from infinity back to the point of interest on the surface of the sphere. We assume a phase advance for the central source so that its field is in phase with the surface sources. We obtain significantly different Q factors for these two cases.

To find the Q values we first calculate the power radiated at each point of the sphere as the current density times the quadrature phase component of the local potential. The stored energy is the current density times the in-phase component of the local potential and this is derived directly from the measured or calculated surface inductance. The quadrature phase component of the local potential is an induced component from every other point on the sphere and this is well approximated by assuming an equivalent central source for this, with a distance time delay. We also have to take into account a non-unity coupling factor $k_c < 1$. Measurements and calculation of the coupling factor of a pair of parallel long thin wires indicate an asymptotic value of about $1/2\pi$. Thus the Q is taken to be 2π times the ratio of reactive to radiated power, derived as 2π times the ratio of in-phase component of the potential divided by its quadrature phase component (relative to the phase of the current). The $1/2\pi$ assumption gives closest agreement with the measurements taken so far [7]. It increases the Q predicted from the potential real and imaginary parts by 2π .

It is the $1/2\pi$ factor that allows the power radiated per unit area at each point of the sphere to be found. It also implies as a consequence that in an untuned antenna any oscillating current (times length) actually creates a near-field and far-field that is less than given by the traditional vector potential method by $1/2\pi$ or -16 dB. By reciprocity this also means that the sensitivity as a field sensor is reduced by $1/2\pi$ or -16 dB. However if the antenna (is low-loss and) is resonated, as for example in a tuned transformer, we find that the potential in which the current elements sit is increased exactly by the amount required to compensate for the $1/2\pi$ value of sensitivity and we end up with the traditional current-to-far-field 'sensitivity' factor extensively used in antenna field calculations.

In case (a), example 2, we use the traditional vector potential radial distance factor e^{-jkr}/r for the surface currents on the sphere. De Moivre's theorem expands this to be:

$$\begin{aligned} D(r) &= \frac{e^{-jkr}}{r} = r^{-1} \cos(kr) - jr^{-1} \sin(kr) \\ &= -y_0(kr) - jj_0(kr) \end{aligned} \quad (4)$$

where $y_0(kr)$ and $j_0(kr)$ are half-order Bessel functions [8]. Thus the Q is:

$$\begin{aligned} Q_2 &= \frac{2\pi \operatorname{Re}[\Phi_2]}{\operatorname{Im}[\Phi_2]} = 2\pi \cot ka, \text{ and for small } ka, \\ &= \frac{6.28}{(ka)} = \frac{\lambda}{a} \end{aligned} \quad (5)$$

where Φ_2 is the potential. Note that this is a factor 2π (6.28) more stringent than Grimes and Grimes [6].

For case (b), example 3, the distance factor $D(r)$ of the field is assumed to be spherically symmetrical and to obey the Biot-Savart law with a retardation factor included. We also include a source phase shift $kb = ka$ to give the distance factor as $D(r) = r^{-2} e^{-jk(r-a)}$. We have assumed that the equivalent source at the centre of the sphere is phase advanced so that its field is in phase with the current on the surface of

the sphere. The potential Φ_3 on the surface of the sphere is then found as follows (Abramovitz and Stegun [8]):

$$\begin{aligned}\Phi_3 &= \Phi_0 \int_{\infty}^a r^{-2} e^{-jk(r-a)} dr \\ &\cong \Phi_0 a^{-1} \left[\frac{\pi ka}{2} - 1 + jka \{-\gamma - ka \ln(ka)\} \right] \quad (6)\end{aligned}$$

γ is Euler's constant, 0.57721566, $b=a$, and we have used series expansions of the resulting sine and cosine integrals that apply for $ka < 1$. Thus we have

$$Q_3 = \frac{2\pi\{\pi ka/2 - 1\}}{ka\{-\gamma - \ln(ka)\}} \cong \frac{6.28}{ka \ln(1.78ka)} \quad (7)$$

This Q criterion is logarithmically more benign than $Q_2 = 6.28/(ka) = \lambda/a$.

Example 4—Line integral potential(s) for a loop: The preceding potential method can be extended down to the conductor surface of a (tuned) loop to find the radiated power per unit length.

To find the potential we integrate the gradient of the magnetic field, as given by the Biot-Savart law, at right angles to the field direction along the line from infinity joining the source element to the point of interest. The gradient of the field is r^{-2} , because it represents the force exerted on a parallel unit current element at the point of interest. For elements that are at an angle θ the force law is $\{\cos(1/2\theta)/r\}^2$. θ is also the radial angle subtended at the centre of the loop and r is the distance from one point on the loop to another point on the loop, so that $r = 2a [\cos(1/2\theta)]$. The coupling of $1/2\pi$ reduces the force law by $1/2\pi$. Thus the coupled line-integral potential contribution of all current elements $i dl = ia d\theta$ around the loop of radius a is

$$\begin{aligned}\frac{\Phi}{2\pi} &= \frac{1}{2\pi} \int_0^{2\pi} \cos^2\theta \left[\int_{\infty}^r r^{-2} e^{-jkr} dr \right] ia d\theta \\ &\cong \frac{i}{2\pi} \int_0^{2\pi} \cos^2\theta \left[\frac{1}{2} \pi k - r^{-1} + jk\{1 - \gamma - \ln(kr)\} \right] a d\theta \quad (8)\end{aligned}$$

To find the loop series radiation resistance R , we integrate around the circumference $2\pi a$ to give

$$\begin{aligned}V &= iR = 2\pi a \text{Im} \left(\frac{\Phi}{2\pi} \right) = ia^2 \int_0^{2\pi} \cos^2 \left(\frac{1}{2} \theta \right) \\ &\quad \times k \left[1 - \gamma - \ln \left\{ 2ka \cos \left(\frac{1}{2} \theta \right) \right\} \right] d\theta \\ &= ika^2 \left\{ \pi(1 - \gamma) - \frac{\pi}{2} - \pi \ln(2ka) \right\} \quad (9) \\ \therefore R &= \pi ka^2 \{0.5 - \gamma - \ln(2ka)\} \\ &= -\pi ka^2 \ln(1.851ka) = -2\pi^2 \frac{a^2}{\lambda} \ln \left(\frac{11.63a}{\lambda} \right) \quad (10)\end{aligned}$$

For convenience, we choose a loop conductor size to give a specific inductance of $L_1 = 1 \mu\text{H/m}$ then

$$Q_4 = \omega L_1 \times \frac{2\pi a}{R} = \frac{2\pi a \omega L_1}{\{-\pi ka^2 \ln(1.851ka)\}} = \frac{600}{\{-a \ln(1.851ka)\}} \quad (11)$$

The logarithmic factor gives a weak dependence on frequency, and refines equation 4 of [7]. The dominant factor determining the Q of a loop is a^{-1} .

The tuned loop dipole mode Q can be estimated from example 1 by the observation that the inductance of a loop of radius a is approximately doubled over the dipole conductor of length $2a$. The dipole mode and loop mode radiation resistances are equal if $2Q_1 = Q_4$ when $(2 \times 7.5)/(ka[-\ln(ka) = 600/\{-a \ln(1.851ka)\}])$ or $\lambda \cong 64$ m. This predicts that the frequency at which a loop in free space is unidirectional [9] is approximately 4.7 MHz. (3.8 MHz has been measured, at 6 metres above ground.)

Example 5—Small dipole by line-integral potential: For brevity we start with the text book approach that gives the radiation resistance as $R_{dip} \cong 20(ka)^2$. We argue that had we applied the line-integral method, we would have obtained the asymptotic logarithmic factor ' $-\ln(ka)$ ' both to increase the radiation resistance R_{dip} and decrease the 'traditional' $Q \cong (ka)^{-3}$. We therefore propose that

$$R_{dip} \cong 20(ka)^2 [-\ln(ka)] \quad (12)$$

$$Q_5 \cong (ka)^{-3} [-\ln(ka)]^{-1} \quad (13)$$

Together with example 4, this shows that any field sensors should not only take into account any coupling loss but should also use this logarithmic correction; otherwise they should be calibrated over the required frequency range.

Acknowledgment: M. Harper has an EPSRC studentship with dstl, Fort Halstead.

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22 April 2003

Electronics Letters Online No: 20030540

DOI: 10.1049/el:20030540

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