ARTICLE IN PRESS
YJMAA:17913

J. Math. Anal. Appl. ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Explicit estimates on the torus for the sup-norm and the dissipative length scale of solutions of the Swift–Hohenberg Equation in one and two space dimensions

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ARTICLE INFO

Article history:
Received 12 April 2013
Available online xxxx
Submitted by Richard M. Aron

Keywords:
Dissipative partial differential equations
Interpolation inequalities
Best constants
Analysis of solutions
Dissipative length scale

ABSTRACT

In this work we have obtained explicit and accurate estimates of the sup-norm for solutions of the Swift–Hohenberg Equation (SHE) in one and two space dimensions. By using the best (so far) available estimates of the embedding constants which appear in the classical functional interpolation inequalities used in the study of solutions of dissipative partial differential equations, we have evaluated in an explicit manner the values of the sup-norm of the solutions of the SHE. In addition we have calculated the so-called time-averaged dissipative length scale associated to the above solutions.

1. Introduction

Partial differential equations (PDEs) represent one of the best ways to mathematically describe important features of many real world phenomena. Thus it is imperative to endeavour to obtain explicit and accurate estimates of their solutions. This is generally achieved by employing a variety of recent results which improve the mathematical techniques used in the analysis of the solutions of PDEs. For instance in this work we use the latest explicit and sharp estimates for the embedding constants appearing in the functional inequalities applied in the study of any PDE. This represents an important step-forward for having, in particular, explicit and sharp estimates on various Sobolev norms associated to the solutions of PDEs including the sup-norm. It is clear that such a result is genuinely indispensable if one wishes to have detailed accuracy in the huge amount of numerical works carried out all over the world for simulating the solutions of dissipative PDEs. Additionally, the sharp and explicit estimate of the embedding constants also shed light on another essential ingredient in the analysis of dissipative PDEs, namely the so-called length scales present in the flow, including the famous dissipative length scale which provides very important information about the smallest features in the flow, and therefore its accurate estimate is essential for a detailed and reliable numerical investigation of the solutions of any dissipative PDE. The dissipative length scale also give very useful information on the patterns involved in the dynamical flows of any dissipative PDE, and are one of the most important dynamical concepts for properly understanding the spatio-temporal patterns of dissipative flows. In addition the dissipative length scale provides a powerful tool for discerning between the so-called soft and hard turbulence regimes in dissipative flows; more precisely given any solution of a dissipative PDE, then if the spatial and temporal average of this solution and its maximum amplitude are of the same order of magnitude, then large fluctuations away from the average are impossible, and thus we are in a soft regime. However, when the maximum amplitude becomes much larger with respect to the average, then the solution has major excursions in space and time; these strong intermittent fluctuations away from the averages are one of the main signatures of hard turbulence. This phenomenon is now well established in many physical contexts such as, for example, in fluid convection. One of the results of this work is the computation of the

0022-247X/$ – see front matter
http://dx.doi.org/10.1016/j.jmaa.2013.09.027

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so-called “crest factor” for the solution of the SHE; the crest factor is precisely the ratio between the L^∞ and the L^2 norm of the solution of any dissipative PDE and hence it addresses the topics mentioned above.

Thus the aim of this work is to tackle some of the above questions for an important dissipative partial differential equation, namely the SHE [7,17,20,21,23,24,27]. It appears in various physical contexts including convective instabilities, nonlinear optic, population dynamics and many others. The SHE can be written in different versions depending on the situation under investigation. Here we will study the following model:

$$u_t = -\Delta^2 u - \Delta u + \lambda u - u^3 \tag{1}$$

for $x \in \Omega = [0, L]^d$, with $L > 0$ and $t > 0$, subject to the initial condition $u(x, 0) = u_0(x)$ and periodic boundary conditions on the boundary of Ω. The parameter λ is a real constant.

Eq. (1) has been investigated by many authors and there are a large number of research papers devoted to the study of various properties of its solutions (see for example [10,20,21]). The layout and main results of the paper are as follows: in Section 2 we state some standard functional setting and the notation used in this work. In Section 3 we obtain explicit and accurate estimates for the sup-norm of the solutions of the SHE in one and two spatial dimensions. These estimates are stated after proving Lemmas 1, 2, 3 and Theorem 1. In Section 4 we compute the time averaged dissipative length scale also in one and two spatial dimensions. Finally in Section 5 we obtain the “crest factor” of the solutions of the SHE and we express the conclusion and open problems.

2. Functional settings and notation

Let us first give some standard preliminary functional setting and notation [1,18,25,28]. Denote by $\Omega = [0, L]^d$ the d-dimensional torus; for any scalar and mean-zero function $\phi(x)$ with $x \in \Omega$ let $\|\phi\|_p^p = \int_\Omega |\phi(x)|^p\,dx$ be the norm associated to the Banach space of Ω-periodic functions; we also define the L^∞ norm as

$$\big\|\phi(x)\big\|_\infty = \sup_{x\in\Omega}\big|\phi(x)\big|. \tag{2}$$

For $p = 2$ we denote by $L^2(\Omega)$ the Hilbert space of Ω-periodic functions; given $n = n_1 + n_2 + \cdots + n_d$ with all the n_i non-negative integers, let

$$D^n := D^{n_1,n_2,\ldots,n_d} = \frac{\partial^{n_1+n_2+\cdots+n_d}}{\partial x_1^{n_1}\partial x_2^{n_2}\cdots\partial x_d^{n_d}}, \tag{3}$$

and let

$$\dot H^n := \left\{\phi \;\Big|\; \int_\Omega \phi\,dx = 0,\ \int_\Omega \left(D^n\phi\right)^2 dx < +\infty \quad \text{for } n_1 + n_2 + \cdots + n_d = n\right\} \tag{4}$$

together with

$$\|\phi\|_{\dot H^n}^2 := \sum_{n=n_1+\cdots+n_d} \frac{n!}{n_1!\cdots n_d!}\|D^n\phi\|_{L^2}^2, \tag{5}$$

be the Sobolev space of mean-zero Ω-periodic functions with up to n-derivatives in $L^2(\Omega)$; in formula (5), for $d = 2$ we naturally identify the functions having the same “mixed” partial derivatives, because it is well known that the solutions of the SHE are sufficiently smooth [2,25,28]; for example we identify the differential operators

$$\frac{\partial^{n_1+n_2+\cdots+n_d}}{\partial x_i^{n_1}\partial x_j^{n_2}\cdots\partial x_d^{n_d}} \equiv \frac{\partial^{n_1+n_2\cdots n_d}}{\partial x_j^{n_1}\partial x_i^{n_2}\cdots\partial x_d^{n_d}}, \tag{6}$$

and of course any other possible combination of the indices. Also from Parseval’s identity we have that

$$\sum_{n=n_1+\cdots+n_d} \frac{n!}{n_1!\cdots n_d!}\big\|D^n\phi\big\|_{L^2}^2 = L^d\left(\frac{2\pi}{L}\right)^{2n} \sum_{n=n_1+\cdots+n_d}\ \sum_{\vec k\in\mathbb{Z}^d\setminus\{\vec 0\}} |\vec k|^{2n}|\phi_{\vec k}|^2. \tag{7}$$

In (7) the Fourier series expansion has been used for the mean-zero function

$$\phi = \sum_{\vec k\in\mathbb{Z}^d\setminus\{\vec 0\}} \phi_{\vec k} e^{2\pi i\vec k\cdot\vec x/L}, \tag{8}$$

and

$$\left(\frac{2\pi}{L}\right)^2 \vec{k}\cdot\vec{k} = \left(\frac{2\pi}{L}\right)^2 \left(k_1^2 + k_2^2 + \cdots + k_d^2\right). \tag{9}$$

By the same token the corresponding Sobolev space of mean-zero periodic functions can be defined as $\dot{H}^s$ for every *real* number s; this is the same as

$$\dot{H}^s = \left\{\phi\colon\ \phi = \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} \phi_{\vec{k}} e^{2\pi i\vec{k}\cdot\vec{x}/L},\ \bar{\phi}_{\vec{k}} = \phi_{-\vec{k}},\ \left(\frac{2\pi}{L}\right)^{2s} \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^{2s}|\phi_{\vec{k}}|^2 < +\infty\right\}. \tag{10}$$

Hence by extending (5) to non-integer positive values we have

$$\dot{H}^s = \left\{\phi\colon\ \|\phi\|^2_{\dot{H}^s} < +\infty\right\}. \tag{11}$$

These Sobolev spaces, defined on the d-dimensional torus, are used below as we need to deal with the negative Laplacian $A = -\Delta$ (as a self-adjoint unbounded operator) and its fractional powers. More precisely we have the eigenvalues of the negative Laplacian $A = -\Delta$ are given by the numbers $(\frac{2\pi}{L})^2|\vec{k}|^2$, so the domain of its powers A^s is the set of functions such that

$$L^d\left(\frac{2\pi}{L}\right)^{4s} \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^{4s}|\phi_{\vec{k}}|^2 = \left\|A^s\phi(x)\right\|_2^2 < +\infty. \tag{12}$$

In particular, for $s = \frac{1}{2}$ (on the torus) we have

$$\left\|A^{\frac{1}{2}}\phi(x)\right\|_2^2 = \left\|\nabla\phi(x)\right\|_2^2 = L^d\left(\frac{2\pi}{L}\right)^2 \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^2|\phi_{\vec{k}}|^2, \tag{13}$$

while for $s = 1$ we have (on the torus)

$$\left\|A\phi(x)\right\|_2^2 = \left\|(-\Delta)\phi(x)\right\|_2^2 = L^d\left(\frac{2\pi}{L}\right)^4 \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^4|\phi_{\vec{k}}|^2. \tag{14}$$

In the rest of the paper (with a minor abuse of notation), for any $s > 0$, we make the *formal* identification

$$\left\|A^{\frac{s}{2}}\phi(x)\right\|_2^2 = \left\|(-\Delta)^{\frac{s}{2}}\phi(x)\right\|_2^2 = L^d\left(\frac{2\pi}{L}\right)^{2s} \sum_{\vec{k}\in\mathbb{Z}^d\setminus\{\vec{0}\}} |\vec{k}|^{2s}|\phi_{\vec{k}}|^2, \tag{15}$$

provided it is understood that these operators are being used as differential operators "acting" on functions in $\dot{H}^s$, according to (10) and (12).

3. Explicit estimates for the L^∞ norm of the solutions of the SHE

In this section we wish to obtain explicit (and as accurately as we possibly can) estimates for the L^∞ norm of the solutions of the SHE. Before beginning let us make clear the notations we will use in this section: because we are going to deal with powers of functions in the Sobolev space

$$\dot{H}^n := \left\{u \;\middle|\; \int_\Omega u\,dx = 0,\ \int_\Omega \left(D^n u\right)^2 dx < +\infty \quad \text{where } n = n_1 + n_2 + \cdots + n_d\right\}, \tag{16}$$

we will define

$$H_n := \|u\|^2_{\dot{H}^n} = \sum_{n=n_1+\cdots+n_d} \frac{n!}{n_1!\cdots n_d!}\left\|D^n u\right\|^2_{L^2}, \tag{17}$$

and as usual we also use $\|u(x,\cdot)\|_\infty = \sup_{x\in\Omega}|u(x,\cdot)|$. Our equation has been defined above and we rewrite it here for the convenience of the reader

$$u_t = -\Delta^2 u - \Delta u + \lambda u - u^3, \tag{18}$$

where Δ is the Laplacian and λ is constant and positive, in the domain $\Omega = [0, L]^d$ with periodic boundary condition. First of all we recall that the SHE is a gradient system [25,26]; therefore it has an associated Lyapunov functional given by

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$$J(u) := \frac{1}{L^d} \int_{\Omega} \left[\frac{1}{2}(\Delta u)^2 - \frac{1}{2}(\nabla u)^2 - \frac{\lambda}{2}u^2 + \frac{1}{4}u^4 \right] dx. \tag{19}$$

This means that the stationary solutions of the problem associated to the SHE are isolated, and any solution tends to one of these stationary states as time goes to plus infinity. Thus Eq. (18) is known to have a unique solution for every initial datum $u_0 \in L_2(0, L)$; the solution $u \in C([0, T]; H)$ where $H = L_2(0, L)$, and $T > 0$; in addition the corresponding semigroup $S_t u_0 = u(t)$ has a global attractor $\mathcal{A} \Subset H$ (for details see [2,28]). Therefore all the calculations and estimates obtained below are *not formal*, but they reflect the actual behavior of the solutions of the SHE. Hence in the following we wish to find as accurately as possible estimates for the H_n and then use them to obtain the corresponding estimates for the L^∞ norm of the solutions by using the sharp estimate found in [3,4,8] (see also [13,19,29,30]).

3.1. Analysis in the one spatial-dimensional case

We can now start our analysis of the SHE on the torus in $d = 1, 2$ spatial dimensions. We begin with the case $d = 1$. First note that one can show that the time-dependent functionals H_n satisfy a so-called *ladder differential inequality* [5,6,12], namely for any $n > \frac{d}{2}$, where d is the spatial dimension, we have that

$$\frac{1}{2}\dot{H}_n \leqslant -H_{n+2} + H_{n+1} + \lambda H_n + c_n \|u\|_\infty^2 H_n, \tag{20}$$

where the constants c_n do not depend upon the solution function $u(x, t)$. Because we need to know explicitly all the constants appearing in our analysis, we are somehow forced to restrict ourselves to the lower values of the non-negative integer n. In particular in the one-dimensional case we can restrict ourselves to the analysis of H_0 and H_1, which together are sufficient for having an upper bound on the $\|u\|_\infty$ norm of the solution of any PDE (in one spatial dimension). On the other hand for the $d = 2, 3$ case we will have to analyze H_2 also. Before starting our formal analysis let us make clear what we mean by the time-asymptotic behavior of a given function of time $F(t)$: From now on with an overbar over a given function of time $F(t)$, namely $\overline{F(t)}$, we mean the so-called limit superior taken over all the initial conditions as time goes to plus infinity; more formally we mean that we are using the classical Gronwall inequality and then we take the limit over all the initial conditions as time goes to infinity; occasionally the set of initial conditions may be restricted to the global attractor of the PDE under investigation, but this will be clear from the context if not explicitly stated [2,25,28].

In space dimension one it is sufficient to have control on the H_0 and the H_1 in order to have control on the sup-norm of any solution of any PDE. Thus we start with the analysis of $H_0(t)$:

Lemma 1. *The time-asymptotic behavior of $H_0(t)$, namely $\overline{H}_0$, is given by*

$$\overline{H}_0 := \lim_{t\to\infty} H_0(t) \leqslant L\left(\lambda + \frac{1}{4}\right). \tag{21}$$

Proof. By taking the time-dependent quantity $H_0(t) = \int u^2(x, t)\, dx$ (to simplify the notation, in this work all the integrals are evaluated on the domain $\Omega = [0, L]^d$, unless otherwise stated) and differentiating it with respect to time one finds

$$\frac{1}{2}\dot{H}_0 = -H_2 + H_1 + \lambda H_0 - \int_{\Omega} (u)^4\, dx. \tag{22}$$

First of all, for non-trivial behavior one can see that we must have a restriction on the values of the parameter λ; in fact, after splitting the H_1 term by using first a Cauchy–Schwarz inequality and then a Young inequality, namely

$$H_1 \leqslant (H_2)^{\frac{1}{2}}(H_0)^{\frac{1}{2}} = (2H_2)^{\frac{1}{2}}\left(\frac{H_0}{2}\right)^{\frac{1}{2}} \leqslant H_2 + \frac{1}{4}H_0,$$

and also noting that $-\int_\Omega (u)^4\, dx \leqslant -\frac{H_0^2}{L}$, it follows that (22) becomes

$$\frac{1}{2}\dot{H}_0 \leqslant \left(\lambda + \frac{1}{4}\right)H_0 - \frac{H_0^2}{L}; \tag{23}$$

hence one can see that if $\lambda \leqslant -\frac{1}{4}$ the zero solution becomes a global attractor (apart of course the two uniform solutions of (18) given by $u = \pm\sqrt{\lambda}$). Thus for non-trivial behavior we need $\lambda > -\frac{1}{4}$ and so to simplify the notation a bit we will take λ to be a positive constant. Thus going back to our analysis of H_0 we have

$$\frac{1}{2}\dot{H}_0 \leqslant \left(\lambda + \frac{1}{4}\right)H_0 - \frac{H_0^2}{L}. \tag{24}$$

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By standard analysis one can see that the fixed points of the corresponding nonlinear ordinary differential equation are given by $H_0 = 0, L(\lambda + \frac{1}{4})$ with 0 being unstable and $L(\lambda + \frac{1}{4})$ being stable. Thus the long-time-asymptotic behavior of H_0 (denoted with $\overline{H}_0$) is given by

$$\overline{H}_0 := \lim_{t\to\infty} H_0(t) \leqslant L\left(\lambda + \frac{1}{4}\right), \tag{25}$$

which is independent of the initial condition $u(x, t = 0) = u_0(x)$. □

We now estimate H_1 with a similar strategy.

Lemma 2. *The time-asymptotic behavior of $H_1(t)$, namely $\overline{H}_1$, is given by*

$$\overline{H}_1 := \lim_{t\to\infty} H_1(t) \leqslant \sqrt{2\lambda + 1}L\left(\lambda + \frac{1}{4}\right). \tag{26}$$

Proof. Here we take the time-dependent quantity $H_1(t) = \int (u_x(x, t))^2\, dx$ and differentiating it with respect to time we find

$$\frac{1}{2}\dot{H}_1 = -H_3 + H_2 + \lambda H_1 - 3\int_\Omega u^2 (u_x)^2\, dx; \tag{27}$$

the last term is negative definite for non-zero solutions and so it can be neglected. So we obtain the estimate

$$\frac{1}{2}\dot{H}_1 \leqslant -\frac{H_3}{2} + \frac{(2\lambda + 1)}{2} H_1, \tag{28}$$

where we have used again the Cauchy–Schwarz inequality on the H_2 term, $H_2 \leqslant H_3^{\frac{1}{2}} H_1^{\frac{1}{2}}$, and then a Young inequality. By using now the inequality [5,6,12]

$$H_p \leqslant H_{p+r}^{\frac{q}{r+q}} H_{p-q}^{\frac{r}{r+q}} \tag{29}$$

with $p = 1, r = 2, q = 1$ one obtains $-H_3 \leqslant -\frac{H_1^3}{H_0^2}$. Hence inserting these into (28) and performing a similar analysis to that used in obtaining the estimate (25) one obtains the result

$$\overline{H}_1 := \lim_{t\to\infty} H_1(t) \leqslant (\sqrt{2\lambda + 1})\overline{H}_0 \leqslant \sqrt{2\lambda + 1}L\left(\lambda + \frac{1}{4}\right). \quad \square \tag{30}$$

We are then ready to obtain the estimate for the $\|u(x, t)\|_\infty$ of the solution in the $d = 1$ case by applying the sharp results found in [3,4,8]:

$$\|u(x)\|_\infty \leqslant \left(\frac{\zeta(1 + \epsilon)}{\pi}\right)^{\frac{1}{2}} \left\|(-\Delta)^{\frac{1+\epsilon}{4}} u(x)\right\|_2 + L^{-\frac{1}{2}} H_0^{\frac{1}{2}}, \tag{31}$$

where $\epsilon > 0$ and

$$\zeta(1 + \epsilon) = \sum_{n\geqslant 1} \frac{1}{n^{1+\epsilon}} \tag{32}$$

is the Riemann zeta-function. By taking the value $\epsilon = 1$ we therefore obtain

$$\|u(x)\|_\infty \leqslant \sqrt{\frac{\pi}{6}} \|\nabla u(x)\|_2 + L^{-\frac{1}{2}} H_0^{\frac{1}{2}} = \sqrt{\frac{\pi}{6}} H_1^{\frac{1}{2}} + L^{-\frac{1}{2}} H_0^{\frac{1}{2}}; \tag{33}$$

thus by using (25) and (30) we obtain

$$\overline{\|u(x)\|_\infty} \leqslant \left(\frac{L\pi}{6}\sqrt{2\lambda + 1}\left(\lambda + \frac{1}{4}\right)\right)^{\frac{1}{2}} + \frac{\sqrt{4\lambda + 1}}{2}. \tag{34}$$

3.2. Analysis in the two spatial dimensions case

We can now turn our attention to the two-dimensional case having domain $[0, L]^2$; as it is well known in this case having control on the H_1 norm alone is not sufficient, but it is necessary to have control on the H_2 norm as well. Before actually computing the time-asymptotic behavior of H_2 we note that the estimates for $\overline{H}_0$ and $\overline{H}_1$ in two spatial dimension are virtually the same as in the one-dimensional case; indeed the only difference is in taking care of the nonlinear term which for $d = 2$ yields $-\int_\Omega (u(x, y, t))^4\,dx\,dy \leqslant -\frac{H_0^2}{L^2}$; it follows that the differential inequality for $H_0(t)$ becomes

$$\frac{1}{2}\dot{H}_0 \leqslant \left(\lambda + \frac{1}{4}\right)H_0 - \frac{H_0^2}{L^2}. \tag{35}$$

Therefore one obtains for the time-asymptotic behavior of $H_0(t)$ the estimate

$$\overline{H}_0 := \lim_{t\to\infty} H_0(t) \leqslant L^2\left(\lambda + \frac{1}{4}\right). \tag{36}$$

Similarly for the time-asymptotic behavior of H_1 one finds that

$$\frac{1}{2}\dot{H}_1 = -H_3 + H_2 + \lambda H_1 - \sum_{n=n_1+n_2=1} \int_\Omega 3u^2(Du)^2\,dx\,dy; \tag{37}$$

hence by neglecting the last two terms in the summation one finally obtains

$$\overline{H}_1 := \lim_{t\to\infty} H_1(t) \leqslant (\sqrt{2\lambda+1})\overline{H}_0 \leqslant \sqrt{2\lambda+1}L^2\left(\lambda + \frac{1}{4}\right). \tag{38}$$

We now turn our attention to the analysis of $H_2(t)$; the corresponding first order nonlinear differential equation is given by

$$\frac{1}{2}\dot{H}_2 = -H_4 + H_3 + \lambda H_2 - \sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_\Omega \left(D^2u\right)\left(D^2u^3\right)dx\,dy, \tag{39}$$

where the terms in the summation represent the nonlinear term. A detailed and thorough analysis of this term has been already performed in [3] but we report it here for the convenience of the reader:

Lemma 3. *The nonlinear term above obeys the estimate*

$$-\sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_\Omega \left(D^2u\right)\left(D^2u^3\right)dx\,dy \leqslant \frac{78}{\pi}H_1H_2. \tag{40}$$

Proof. One starts by making the explicit differentiation, thereby obtaining

$$-\sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_\Omega \left(D^2u\right)\left(D^2u^3\right)dx\,dy \tag{41}$$

$$= -6\int_\Omega u(u_x)^2u_{xx} - 3\int_\Omega u^2(u_{xx})^2\,dx\,dy - 6\int_\Omega u(u_y)^2u_{yy}\,dx\,dy - 3\int_\Omega u^2(u_{yy})^2\,dx\,dy \tag{42}$$

$$-6\int_\Omega u^2(u_{xy})^2\,dx\,dy - 12\int_\Omega uu_xu_yu_{xy}\,dx\,dy; \tag{43}$$

integrating by parts the first, the third and the last terms and then rearranging we obtain

$$-\sum_{n=n_1+n_2=2} \frac{2!}{n_1!n_2!} \int_\Omega \left(D^2u\right)\left(D^2u^3\right)dx\,dy \tag{44}$$

$$= 2\int_\Omega (u_x)^4\,dx\,dy - 3\int_\Omega u^2(u_{xx})^2\,dx\,dy + 2\int_\Omega (u_y)^4\,dx\,dy - 3\int_\Omega u^2(u_{yy})^2\,dx\,dy \tag{45}$$

$$-6\int_\Omega u^2(u_{xy})^2\,dx\,dy + 6\int_\Omega (u_x)^2(u_y)^2\,dx\,dy + 6\int_\Omega uu_{xx}(u_y)^2\,dx\,dy; \tag{46}$$

by splitting the last two terms by applying first a Cauchy–Schwarz inequality and then a Young inequality we get

$$-\sum_{n=n_1+n_2=2}\frac{2!}{n_1!n_2!}\int_\Omega (D^2u)(D^2u^3)\,dx\,dy \tag{47}$$

$$=2\int_\Omega (u_x)^4\,dx\,dy-3\int_\Omega u^2(u_{xx})^2\,dx\,dy+2\int_\Omega (u_y)^4\,dx\,dy-3\int_\Omega u^2(u_{yy})^2\,dx\,dy \tag{48}$$

$$-6\int_\Omega u^2(u_{xy})^2\,dx\,dy+3\int_\Omega (u_x)^4\,dx\,dy+3\int_\Omega (u_y)^4\,dx\,dy \tag{49}$$

$$+3\int_\Omega u^2(u_{xx})^2\,dx\,dy+3\int_\Omega (u_y)^4\,dx\,dy; \tag{50}$$

simplifying we finally obtain that the nonlinear term can be estimated as follows:

$$-\sum_{n=n_1+n_2=2}\frac{2!}{n_1!n_2!}\int_\Omega (D^2u)(D^2u^3)\,dx\,dy\leqslant 5\int_\Omega (u_x)^4\,dx\,dy+8\int_\Omega (u_y)^4\,dx\,dy. \tag{51}$$

Thus we have to estimate the terms $5\int_\Omega (u_x)^4\,dx\,dy$ and $8\int_\Omega (u_y)^4\,dx\,dy$. In the two-dimensional case we can use an improved version of the Ladyzhenskaya inequality [15], namely for any mean-zero function $\phi(x,y)$ on the 2d torus we have the inequality

$$\int_\Omega (\phi(x,y))^4\,dx\,dy\leqslant\frac{6}{\pi}\int_\Omega (\phi(x,y))^2\,dx\,dy\int_\Omega |\nabla\phi|^2\,dx\,dy;$$

hence we can estimate the term $5\int_\Omega (u_x)^4\,dx\,dy$ as

$$5\int_\Omega (u_x)^4\,dx\,dy\leqslant\frac{30}{\pi}\left(\int_\Omega (u_x)^2\,dx\,dy\right)\left(\int_\Omega (u_{xx}^2+u_{xy}^2)\,dx\,dy\right)$$

and similarly

$$8\int_\Omega (u_y)^4\,dx\,dy\leqslant\frac{48}{\pi}\left(\int_\Omega (u_y)^2\,dx\,dy\right)\left(\int_\Omega (u_{yy}^2+u_{xy}^2)\,dx\,dy\right).$$

By noting that $\int_\Omega (u_x)^2\,dx\,dy\leqslant H_1$, $\int_\Omega (u_y)^2\,dx\,dy\leqslant H_1$ and $\int_\Omega (u_{xx}^2+u_{xy}^2)\,dx\,dy\leqslant H_2$, $\int_\Omega (u_{yy}^2+u_{xy}^2)\,dx\,dy\leqslant H_2$, we therefore obtains our result which reads

$$-\sum_{n=n_1+n_2=2}\frac{2!}{n_1!n_2!}\int_\Omega (D^2u)(D^2u^3)\,dx\,dy\leqslant\frac{78}{\pi}H_1H_2.\qquad\square \tag{52}$$

By using the results obtained above we can now prove the following

Theorem 1. *The time-asymptotic behavior of $H_2(t)$, namely $\overline{H}_2$, is given by*

$$\overline{H}_2\leqslant\left(2\lambda+1+\frac{156}{\pi}\sqrt{2\lambda+1}L^2\left(\lambda+\frac{1}{4}\right)\right)^{\frac{1}{2}}\sqrt{2\lambda+1}L^2\left(\lambda+\frac{1}{4}\right). \tag{53}$$

Proof. First we write (39) as

$$\frac{1}{2}\dot{H}_2\leqslant -H_4+H_3+\lambda H_2+\frac{78}{\pi}H_1H_2. \tag{54}$$

Doing similar transformations used for obtaining the asymptotic behavior of H_1 and H_2 we arrive at the inequality

$$\frac{1}{2}\dot{H}_2\leqslant -\frac{1}{2}H_4+\left(\lambda+\frac{1}{2}+\frac{78}{\pi}H_1\right)H_2; \tag{55}$$

we now use the inequality [5,6,12] $H_p\leqslant H_{p+r}^{\frac{q}{r+q}}H_{p-q}^{\frac{r}{r+q}}$ with $p=2, r=2, q=1$ and obtain the time-asymptotic behavior of H_2:

$$\overline{H}_2\leqslant\left(2\lambda+1+\frac{156}{\pi}\overline{H}_1\right)^{\frac{1}{2}}\overline{H}_1. \tag{56}$$

By substituting the estimate for $\overline{H}_1$ we finally obtain

$$\overline{H}_2 \leqslant \left(2\lambda + 1 + \frac{156}{\pi}\sqrt{2\lambda+1}L^2\left(\lambda + \frac{1}{4}\right)\right)^{\frac{1}{2}}\sqrt{2\lambda+1}L^2\left(\lambda+\frac{1}{4}\right). \quad \square \tag{57}$$

Thus for the estimate of $\|u(x)\|_\infty$ we use the result proved in [4], where it is shown that on the two-dimensional torus $\Omega = [0, L]^2$, for every positive real number $s = 1+\epsilon$ with $\epsilon > 0$, the L^∞ norm of a mean-zero scalar function $u(x) \in \dot{H}^{1+\epsilon}$ satisfies the estimate

$$\|u(x)\|_\infty \leqslant \left[4\zeta(1+\epsilon)\beta(1+\epsilon)\right]^{\frac{1}{2}} L^{-1}\left(\frac{L}{2\pi}\right)^{(1+\epsilon)} \left\|(-\Delta)^{\frac{1+\epsilon}{2}}u(x)\right\|_2, \tag{58}$$

where the coefficient $4\zeta(1+\epsilon)\beta(1+\epsilon)$ is sharp, and where

$$\zeta(1+\epsilon) = \sum_{n\geqslant 1}\frac{1}{n^{1+\epsilon}}, \qquad \beta(1+\epsilon) = \sum_{n\geqslant 0}\frac{(-1)^n}{(2n+1)^{1+\epsilon}}, \tag{59}$$

are the Riemann zeta-function and Dirichlet series respectively.

Thus for the estimate of $\overline{\|u(x)\|_\infty}$ we use (58) with $\epsilon = 1$, namely

$$\overline{\|u(x)\|_\infty} \leqslant \frac{L}{2\pi^2}\left(\zeta(2)\beta(2)\right)^{\frac{1}{2}}\left\|\Delta u(x)\right\|_2 + L^{-1}\overline{H}_0^{\frac{1}{2}} \leqslant \frac{L}{2\pi^2}\left(\zeta(2)\beta(2)\right)^{\frac{1}{2}}\overline{H}_2^{\frac{1}{2}} + L^{-1}\overline{H}_0^{\frac{1}{2}}. \tag{60}$$

By using the values for $\zeta(2)\beta(2) = 6\pi^{-2}K$ with $K = 0.915965594\ldots$ we obtain

$$\overline{\|u(x)\|_\infty} \leqslant \frac{L}{2\pi^3}\sqrt{6K}\,\overline{H}_2^{\frac{1}{2}} + L^{-1}\overline{H}_0^{\frac{1}{2}}, \tag{61}$$

where the estimate for $\overline{H}_2^{\frac{1}{2}}$ is provided by (57) and that for $\overline{H}_0^{\frac{1}{2}}$ is provided by (36).

4. Time averaged dissipative length scale

In this section we wish to obtain time-average estimates and then the time averaged length scales for the solutions of our equation. For the definition and computation of the dissipative length scale in dissipative nonlinear PDEs we follow [5,6,12]. So we need to obtain estimates for the time-average of the quotient between the L^∞ norm and the L^2 norm of the solution, namely $\langle \|u\|_\infty / H_0^{\frac{1}{2}}\rangle$. First of all let us derive sharp estimates for the $\|u\|_\infty$ of typical solutions $u(x,t)$. Note that in general we cannot assume that the solutions of the SHE have zero-mean. Hence we have to "carry along" the mean value of our solutions. Thus suppose that $\int_\Omega u(x)\,dx \neq 0$ and $u(x) = u^* + u'(x)$, where $u^* = \text{const} \neq 0$ and $\int_\Omega u'(x)\,dx = 0$. Then using the inequality

$$|u^*| = L^{-d}\left|\int_\Omega u(x)\,dx\right| \leqslant L^{-\frac{d}{2}}H_0^{\frac{1}{2}}$$

we obtain [7]

$$\|u\|_\infty \leqslant \left|u^*\right| + \left\|u'\right\|_\infty \leqslant L^{-\frac{d}{2}}H_0^{\frac{1}{2}} + c(n)H_0^{\frac{2n-d}{4n}}\,H_n^{\frac{d}{4n}} \tag{62}$$

with $n > \frac{1}{2}$, where we have used a Gagliardo–Nirenberg inequality to obtain the estimate $\|u'\|_\infty \leqslant c(n)(\frac{H_n}{H_0})^{\frac{d}{4n}}H_0^{\frac{1}{2}}$. By substituting $u = 1$ in (62) we see that the constant $L^{-\frac{d}{2}}$ is sharp. Therefore we obtain the following estimate

$$\frac{\|u\|_\infty}{H_0^{\frac{1}{2}}} \leqslant \frac{|u^*| + \|u'\|_\infty}{H_0^{\frac{1}{2}}} \leqslant L^{-\frac{d}{2}} + \frac{\|u'\|_\infty}{H_0^{\frac{1}{2}}}. \tag{63}$$

Hence by using (62) we have

$$L^{-\frac{d}{2}} + \frac{\|u'\|_\infty}{H_0^{\frac{1}{2}}} \leqslant L^{-\frac{d}{2}} + c(n)\left(\frac{H_n}{H_0}\right)^{\frac{d}{4n}}.$$

Following [5,6,12] we therefore define our dissipative length scale to be

$$(l)^{-\frac{d}{2}}(t) = c(n)\left(\frac{H_n}{H_0}\right)^{\frac{d}{4n}}. \tag{64}$$

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Formula (64) comes from the Fourier series expansion of any periodic function; it takes into account the fact that the dissipative length scale in a periodic domain of length L in one spatial dimension is given by $l = LN^{-1}$, where N is the nth Fourier mode such that all the higher modes decay in time (in the linearized case). As was shown in [6], the time-dependent quantities (64) effectively define a time-dependent length scale and its time evolution is generally quite difficult to analyze; hence a useful and more manageable quantity would be a time-independent length scale which is normally obtained by taking time averages or asymptotic upper bounds. Here we wish to derive time averaged length scales. So we need to obtain the time average of (64). We first do the one-dimensional case and then the two-dimensional one.

4.1. Time-averaged dissipative length scale in one spatial dimension

In one spatial dimension it is sufficient to take $n = 1$ in (64) and so by taking the square of (64) and time averaging one gets

$$l^{-1} = c^2(1)\left\langle\left(\frac{H_1}{H_0}\right)^{\frac{1}{2}}\right\rangle, \tag{65}$$

where we are denoting the time averaged length scale by dropping its time dependence. Thus one needs to derive as best as possible the time average of the quantity $(\frac{H_1}{H_0})^{\frac{1}{2}}$. This is achieved as follows. First take the differential inequality (28), namely

$$\frac{1}{2}\dot{H}_1 \leqslant -\frac{H_3}{2} + \frac{(2\lambda+1)}{2}H_1; \tag{66}$$

by using (29) with $p = 1, r = 2, q = 1$ one obtains $-H_3 \leqslant -\frac{H_1^3}{H_0^2}$, and so (66) can be recast into

$$\frac{1}{2}\dot{H}_1 \leqslant -\frac{H_1^3}{2H_0^2} + \frac{(2\lambda+1)}{2}H_1. \tag{67}$$

Now first divide throughout by H_1 obtaining

$$\frac{1}{2}\dot{H}_1 \leqslant -\frac{H_1^2}{2H_0^2} + \frac{(2\lambda+1)}{2}. \tag{68}$$

Then we take the time average of both sides of the inequality thereby getting

$$\left\langle\left(\frac{H_1}{H_0}\right)^2\right\rangle \leqslant 2\lambda + 1. \tag{69}$$

Going back to (65) one obtains (with $c^2(1) = 1$, see [14] or Appendix A in [7])

$$l^{-1} = \left\langle\left(\frac{H_1}{H_0}\right)^{\frac{1}{2}}\right\rangle \leqslant \left(\left\langle\left(\frac{H_1}{H_0}\right)^2\right\rangle\right)^{\frac{1}{4}} \leqslant (2\lambda+1)^{\frac{1}{4}}, \quad \text{with } \lambda > 0. \tag{70}$$

One can then compute the fractal dimension of the corresponding global attractor $\mathcal{A}$ using the classical formula (see [9,25,28])

$$\dim_F \mathcal{A} \simeq \frac{1}{l} \leqslant (2\lambda+1)^{\frac{1}{4}}.$$

By using the theory of global attractor dimension a similar estimate was obtained in [7] (formula (4.9) with $\alpha = \beta = 1$).

4.2. Time-averaged dissipative length scale in two spatial dimensions

The strategy for obtaining the time-averaged dissipative length scale in two spatial dimensions is similar to the one-dimensional case with the extra care of inserting the corresponding explicit values of the constants $c(n)$ in (64). Also it is well known that in two spatial dimensions it is sufficient to take $n = 2$, namely we need to estimate the time average of the quotient $\langle(\frac{H_2}{H_0})^{\frac{1}{2}}\rangle$. So we start from the differential inequality (55), namely

$$\frac{1}{2}\dot{H}_2 \leqslant -\frac{1}{2}H_4 + \left(\lambda + \frac{1}{2} + \frac{78}{\pi}H_1\right)H_2; \tag{71}$$

we now use again the inequality [5,6,12] $H_p \leqslant H_{p+r}^{\frac{q}{r+q}} H_{p-q}^{\frac{r}{r+q}}$ with $p = 2, r = 2, q = 2$ and so we obtain

$$\frac{1}{2}\dot{H}_2 \leqslant -\frac{1}{2}\frac{H_2^2}{H_0} + \left(\lambda + \frac{1}{2} + \frac{78}{\pi}H_1\right)H_2. \tag{72}$$

Similarly to the one-dimensional case we divide throughout by H_2 and then we take the time average of both sides of the inequality obtaining

$$\left\langle\frac{H_2}{H_0}\right\rangle \leqslant (2\lambda + 1) + \frac{156}{\pi}\langle H_1\rangle. \tag{73}$$

In order to estimate $\langle H_1\rangle$ we use

$$\frac{1}{2}\dot{H}_0 = -H_2 + H_1 + \lambda H_0 - \int_\Omega (u)^4\,dx. \tag{74}$$

By using $H_1 \leqslant H_2^{\frac{1}{2}} H_0^{\frac{1}{2}}$ and then splitting the right hand side with the Young inequality we obtain

$$\frac{1}{2}\dot{H}_0 = -\frac{H_2}{2} + \left(\lambda + \frac{1}{2}\right)H_0 - \int_\Omega (u)^4\,dx. \tag{75}$$

By neglecting the last term and time averaging both sides we finally get

$$\langle H_2\rangle \leqslant (2\lambda + 1)\langle H_0\rangle. \tag{76}$$

Let us now time average both sides of $H_1 \leqslant H_2^{\frac{1}{2}} H_0^{\frac{1}{2}}$ obtaining

$$\langle H_1\rangle \leqslant \langle H_2\rangle^{\frac{1}{2}}\langle H_0\rangle^{\frac{1}{2}} \leqslant (2\lambda + 1)^{\frac{1}{2}}\langle H_0\rangle \leqslant (2\lambda + 1)^{\frac{1}{2}}\overline{H}_0. \tag{77}$$

In the last inequality we have used the result that in the global attractor it holds that $\langle H_0\rangle \leqslant \overline{H}_0$. We are now ready to compute the dissipative length scale in the two-dimensional case. Take (64) with $d = 2$ and $n = 2$; we obtain

$$l^{-1} = c(2)\left\langle\left(\frac{H_2}{H_0}\right)\right\rangle^{\frac{1}{4}}. \tag{78}$$

Note that we must insert the value of the constant $c(2)$ for $d = 2$ and $n = 2$. Its value is $c(2) = \sqrt{\frac{1}{\pi}}$ (see [16]). Hence just take the estimates (73), (77) and (36) thereby getting the estimate

$$l^{-1} = \sqrt{\frac{1}{\pi}}(2\lambda + 1)^{\frac{1}{8}}\left[(2\lambda + 1)^{\frac{1}{2}} + \frac{156}{\pi}L^2\left(\lambda + \frac{1}{4}\right)\right]^{\frac{1}{4}}. \tag{79}$$

5. Conclusions and open problems

In this work we have analyzed various Sobolev norms of solutions of the SHE, with the aim to estimate as accurately as possible the sup-norm of solutions. More specifically, by using the *best* available explicit estimates for the coefficients which appear in the Sobolev norms used, we have first derived explicit estimates for the $\overline{H}_0, \overline{H}_1, \overline{H}_2$, namely their time-asymptotic behavior, and then we have used these estimates to compute the time-asymptotic behavior of the L^∞ norm of the solution, namely the $\overline{\|u(x)\|_\infty}$ in one and two space dimensions. A very interesting open problem which arises naturally from our analysis is to investigate the so-called *crest factor* (also known as the *peak to average ratio*), namely the ratio between the L^∞ norm of the solution and the L^2 norm of the solution:

$$C_f := \frac{\|u\|_\infty}{H_0^{\frac{1}{2}}}. \tag{80}$$

It has therefore the dimension of the square root of the inverse of the "volume" of the torus in d spatial dimensions, and hence it can be made dimensionless by multiplying (80) by $L^{\frac{d}{2}}$. The crest factor contains important informations on the "distorsions" between the amplitude and the L^2 norm of the solution. It is in fact a standard measurement used in turbulence experiments in fluid dynamics. The ideal result would be to have a time-pointwise estimate of C_f. However this is very difficult due to the nonlinearity of the equation. Alternatively one could try to estimate the time-asymptotic behavior of C_f, but this also proves to be very hard to handle and it is essentially due to the lack of knowledge of a "decent" lower bound on the quantity H_0, namely an estimate of the form $H_0(t) \geqslant \alpha > 0$, where α is a "not too large" positive constant. The problem of estimating the lower bound appears in many contexts in the theory of nonlinear dissipative PDEs, such as for example in the theory of the Navier–Stokes equations where it is notoriously very hard to find a "proper" lower bound for the energy even on the torus [11]. Nevertheless for the SHE one can compute the time-average of the crest factor C_f.

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By taking for simplicity the length $L = 2\pi$ and spatial dimension $d = 1$ (for $d = 2$ one would obtain a similar result) we reason as follows: take (33) which we rewrite here for convenience

$$\|u(x)\|_\infty \leqslant \sqrt{\frac{\pi}{6}} H_1^{\frac{1}{2}} + (2\pi)^{-\frac{1}{2}} H_0^{\frac{1}{2}}; \tag{81}$$

if we divide through by $H_0^{\frac{1}{2}}$ we obtain exactly formula (63) (with $L = 2\pi$), namely

$$\frac{\|u\|_\infty}{H_0^{\frac{1}{2}}} \leqslant \frac{|u^*| + \|u'\|_\infty}{H_0^{\frac{1}{2}}} \leqslant (2\pi)^{-\frac{d}{2}} + \frac{\|u'\|_\infty}{H_0^{\frac{1}{2}}}. \tag{82}$$

Taking now the time average of both sides of (82) one obtains

$$\left\langle \frac{\|u(x)\|_\infty}{H_0^{\frac{1}{2}}} \right\rangle \leqslant \sqrt{\frac{\pi}{6}} (2\lambda + 1)^{\frac{1}{4}} + (2\pi)^{-\frac{1}{2}}. \tag{83}$$

Formula (83) is quite expressive and reveals that the time average of the ratio between the "peak to the root mean square" scales like (83) as a function of the positive parameter λ. So for small λ it is small as it should be, and in general it increases following the estimate (83). This is in agreement with a qualitative analysis of the solutions of the SHE [20,22]. Furthermore (83) reveals the intimate connection between the estimates for the crest factor, the dissipative length scale and the fractal dimension of the global attractor of any dissipative PDE. We believe this strong correlation should be further investigated with the aim to make clearer the relationship among these basic features of any dissipative PDE.

Acknowledgments

It is a pleasure to acknowledge very helpful discussions with Edriss Titi and Stephen Gourley on various features of this work. We wish also to thank the anonymous Referee for her/his suggestions and comments which have helped to improve the content and presentation of this work.

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