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# On a class of Hill's equations having explicit solutions

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ABSTRACT

We present a class of Hill's equations possessing explicit solutions through elementary functions. In addition we provide some applications by using some of the paradigmatic systems of classical dynamics, such as the pendulum with variable length.



## 1. Introduction

Autonomous dynamical systems with one degree of freedom are (essentially) well understood. When the system is non-autonomous, in general the dynamics become unpredictable and the solutions are no longer expressible in "closed form"; for instance in the Hamiltonian case the system ceases to be integrable. However there are some exceptions to the above picture of dynamics. For instance there are very interesting dynamical systems, even with an infinite number of degrees of freedom, which are integrable; a well known example is the class of evolution equations having soliton solutions [1,2].

The aim of this paper is to present and investigate a class of equations having explicit solutions, which can be given in closed form in terms of elementary functions, such as polynomials and trigonometric functions. Our class of equations includes a set of Hill's equations [3], which notoriously have solutions *not* in explicit form, but rather as time series; a very famous example is Mathieu's equation.

First let us recall a brief summary of the results contained in [4] (see also [5]), which are used to define and study the class of ordinary differential equations with variable coefficients possessing explicit solutions. We shall consider a class of non-conservative systems which naturally extend the classical formula for a conservative system with one degree of freedom, namely Newton's formula

$$\dot{x}(t) = y(t), \qquad \dot{y}(t) = -\omega^2 F(x(t)), \tag{1}$$

where $\omega > 0$ is a constant, $F(x)$ is a $C^1$ function of the variable $x$ and the dot denotes derivative with respect to time. If one writes (1) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}(t)}{\omega}\right) + \omega x(t) = 0, \tag{2}$$

the $\omega$ constant case carries over when $\omega = \omega(t)$, that is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}(t)}{\omega(t)}\right) + \omega(t) F(x(t)) = 0. \tag{3}$$

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Indeed the system (3), with $\omega(t)$ a positive differentiable function, has a first integral given by

$$H(x(t), \dot{x}(t), t) := \frac{1}{2}\left(\frac{\dot{x}(t)}{\omega(t)}\right)^2 + U(x(t)) = E = \text{constant}, \tag{4}$$

where $U(x)$ is a primitive of $F(x)$. For instance, for $F(x) = x$, (3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}(t)}{\omega(t)}\right) + \omega(t)\, x(t) = 0, \tag{5}$$

whose general solution has the form

$$x(t) = \alpha \cos\left(\int \omega(t)\, \mathrm{d}t\right). \tag{6}$$

**Remark 1.** Note that (5) can be written in the equivalent form

$$\ddot{x}(t) - \frac{\dot{\omega}(t)}{\omega(t)}\, \dot{x}(t) + \omega^2(t)\, x(t) = 0; \tag{7}$$

this form of the equation will be used below.

More generally, multiplying (4) by $\omega^2(t)$ and doing some rearrangement one obtains

$$\int \frac{\mathrm{d}x}{\sqrt{E - U(x)}} = \pm\sqrt{2}\int \mathrm{d}t\, \omega(t). \tag{8}$$

Formula (8) clearly shows that when $\omega(t)$ is of one sign, then it is effectively a time-reparametrization of the time-independent case. However $\omega(t)$ in general can change sign and still the ratio $\dot{x}(t)/\omega(t)$ is well defined as one can clearly see by looking at formula (6) in the linear case. Thus (6) and (8) naturally extend the case $\omega$ = constant to the case when $\omega = \omega(t)$ is a differential function of time. For more details and applications see [4].

Let us now study the connection between the linear equation (5) and its corresponding "Hamiltonian" form, namely

$$\ddot{y}(t) + Q(t)\, y(t) = 0, \tag{9}$$

where $Q(t)$ is an appropriate function of time. The transformation that takes (7) into (9) is quite general and it can be found (for example) in the book of Magnus and Winkler [3]: the differential equation

$$\ddot{x}(t) + a(t)\, \dot{x}(t) + b(t)\, x(t) = 0, \tag{10}$$

with $a(t)$ and $b(t)$ differentiable functions of $t$, can be transformed into (9), with

$$Q(t) = -\frac{1}{2}\,\dot{a}(t) - \frac{1}{4}\,a^2(t) + b(t) \quad \text{and} \quad y(t) = \left(\mathrm{e}^{\frac{1}{2}\int a(t)\,\mathrm{d}t}\right) x(t). \tag{11}$$

In the case (7) one has

$$a(t) = -\frac{\dot{\omega}(t)}{\omega(t)}, \qquad b(t) = \omega^2(t), \tag{12}$$

so that

$$Q(t) = \frac{1}{2}\frac{\ddot{\omega}(t)}{\omega(t)} - \frac{3}{4}\left(\frac{\dot{\omega}(t)}{\omega(t)}\right)^2 + \omega^2(t) \quad \text{and} \quad y(t) = \frac{x(t)}{\sqrt{\omega(t)}}. \tag{13}$$

It is apparent that in (12) and (13) the function $\omega(t)$ needs to be of one sign without zeros, and if one wants real solutions then of course $\omega(t) > 0$.

## 2. Some equation having explicit solutions

We now provide some illustrations and applications of the results presented above.

### *2.1. (Generalized) Bessel's equation*

Let us consider the linear second order ordinary differential equation

$$\ddot{x}(t) + \frac{\gamma}{t}\,\dot{x}(t) + \frac{t^2 - \nu^2}{t^2}\, x(t) = 0, \quad \nu \geq 0; \tag{14}$$

this equation reduces to the Bessel equation for $\gamma = 1$ [6]. We wish to find explicit solutions of (14) for various values of $\gamma$ and $\nu$ including the Bessel value $\gamma = 1$ (with $\nu = 1/2$). First, for the corresponding transformed Eq. (9) we have

$$Q(t) = \frac{4t^2 - 4\nu^2 - \gamma^2 + 2\gamma}{4t^2}. \tag{15}$$

A solution of (9) is then obtained by putting $Q(t) \equiv 1$, thereby providing the general solution $y(t) = \alpha \cos t + \beta \sin t$. Solving (15) with $Q(t) \equiv 1$ one finds $4\nu^2 = 2\gamma - \gamma^2$, with $0 \le \gamma \le 2$; hence we find the general solution of (14) in the form

$$x(t) = t^{-\frac{\gamma}{2}} \left(\alpha \cos(t) + \beta \sin(t)\right). \tag{16}$$

One recovers the classical solution of the Bessel equation for $\gamma = 1$, requiring the value $\nu^2 = 1/4$. Note that for $\gamma < 0$ and $\gamma > 2$ one can solve Eq. (14) with the plus sign in front of $\nu^2$; in this case one obtains the solution having the same structure as (16) provided one takes $\gamma < 0$ or $\gamma > 2$ and $4\nu^2 = \gamma^2 - 2\gamma$.

### 2.2. *Transformed solutions*

Once the solution to an equation of the form (10) has been found, we can generate a set of equations of which solutions can be found by transformation. For instance, take the two equations

$$\begin{aligned} &\ddot{x}_1(t) + a_1(t)\,\dot{x}_1(t) + b_1(t)\,x_1(t) = 0, \\ &\ddot{x}_2(t) + a_2(t)\,\dot{x}_2(t) + b_2(t)\,x_2(t) = 0, \end{aligned} \tag{17}$$

and assume $a_1(t)$, $b_1(t)$ chosen such that the solution $x_1(t)$ can be found. If $a_2(t)$ and $b_2(t)$ satisfy

$$-\frac{1}{2}\,\dot{a}_1(t) - \frac{1}{4}\,a_1^2(t) + b_1(t) = -\frac{1}{2}\,\dot{a}_2(t) - \frac{1}{4}\,a_2^2(t) + b_2(t), \tag{18}$$

that is $Q_1(t) \equiv Q_2(t)$, the corresponding solution $x_2(t)$ is

$$x_2(t) = x_1(t)\, e^{\frac{1}{2}\int (a_1(t) - a_2(t))\,dt}. \tag{19}$$

As an example with non-constant coefficients, let us take $a_1(t) = \alpha\, t^{-1}$ and $b_1(t) = \beta t^{-2}$, hence picking the Cauchy–Euler equation. In this instance (18) becomes

$$\frac{2\alpha - \alpha^2 + 4\beta}{t^2} = -2\dot{a}_2 - a_2^2(t) + 4b_2(t).$$

Let us take $a_2(t) = \delta t^m$. Hence we require

$$b_2(t) = \frac{\delta^2 t^{2m+2} + 2m\delta t^{m+1} - \alpha^2 + 2\alpha + 4\beta}{4t^2},$$

giving the equation

$$\ddot{x}_2(t) + \delta t^m \dot{x}_2(t) + \frac{\delta^2 t^{2m+2} + 2m\delta t^{m+1} - \alpha^2 + 2\alpha + 4\beta}{4t^2} x_2(t) = 0,$$

which admits the solution

$$x_2(t) = t^{\gamma}\, e^{\frac{1}{2}\int (\alpha t^{-1} - \delta t^m)dt}, \quad \text{with } \gamma = \frac{1}{2}\left(\alpha - 1 \pm \sqrt{(\alpha - 1)^2 - 4\beta}\right).$$

### 2.3. *Pendulum with variable length*

Let us consider the linearized system for a pendulum with periodically varying length $\ell = \ell(t)$ described by,

$$\ddot{\theta}(t) + 2\frac{\dot{\ell}(t)}{\ell(t)}\,\dot{\theta}(t) + \frac{g}{\ell(t)}\,\theta(t) = 0, \tag{20}$$

where $g$ is the gravity acceleration, and take $\ell = \ell_0 + \ell_1 \varphi(\omega t)$, where $\ell_0 > \ell_1 > 0$ and $\varphi$ is a $2\pi$-periodic function of time $\tau = \omega t$ with mean value zero and $\|\varphi\|_\infty = 1$. Hence $\omega$ is the frequency at which the pendulum length varies; for simplicity we shall set $g = 1$. Transforming into the corresponding "Hamiltonian" form we still obtain (9), where

$$Q(t) = \frac{1 - \ddot{\ell}(t)}{\ell(t)}.$$

If we take the particular case where $Q(t) = \omega^2 = \text{constant}$, then we obtain the solution $y(t) = \gamma \cos(\omega t) + \delta \sin(\omega t)$, provided

$$\ell(t) = \alpha \cos(\omega t) + \beta \sin(\omega t) + \frac{1}{\omega^2}, \quad \text{with } \ell_1 = \sqrt{\alpha^2 + \beta^2} < \frac{1}{\omega^2} = \ell_0.$$

Then transforming back to $\theta$ we have a solution to (20) as follows:

$$\theta(t) = \frac{\gamma \cos(\omega t) + \delta \sin(\omega t)}{\alpha \cos(\omega t) + \beta \sin(\omega t) + \omega^{-2}}. \tag{21}$$

**Remark 2.** The choice $Q = \omega^2 = \text{constant}$ effectively sets the mean length of the pendulum $\ell_0 = 1/\omega^2$. In fact the choice of $Q$ equal to any positive constant results in a relation between the class of pendulum systems with periodically varying length around $\ell_0$ with frequency $\omega = 1/\sqrt{\ell_0}$ and the classical pendulum with length $\ell_0$. One way to interpret the result is as follows. Consider the pendulum with variable length $\ell(t) = \ell_0 + \ell_1\varphi(\omega t)$, with fixed frequency $\omega$, in the approximation of small oscillations. The system displays a very complicated behavior for general values of the parameter $\ell_0$; see for instance [7]. However, if one sets $\ell_0 = 1/\omega^2$, then the equations of motion can be explicitly solved.

Some other applications to the pendulum with variable length can be found in [4].

*2.4. Synchronization of the pendulum with variable length and oscillating support*

We may also consider the linearized system for a pendulum with periodically varying length $\ell = \ell(t)$ and oscillating support, so that the motion of the pendulum is described by

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\ell^2(t)\dot{\theta}(t)\Big) + \ell(t)\Big(g - \ddot{\lambda}(t)\Big)\theta(t) = 0 \tag{22}$$

where $\theta(t)$, $\ell(t)$ and $g$ are as in the pendulum with periodically varying length in 2.3 above and the support moves vertically following a given function $\lambda(t)$ of time. For a given function $\lambda(t)$ one can synchronize the time-variable length so that they satisfy the constraint

$$\ell(t) = \left(\frac{1}{g - \ddot{\lambda}(t)}\right)^{\frac{1}{3}}. \tag{23}$$

Then Eq. (22) is of the form (5) and has the solution

$$\theta(t) = \alpha \cos\left(\int \ell(t)\Big(g - \ddot{\lambda}(t)\Big)\mathrm{d}t + \beta\right), \tag{24}$$

for arbitrary constants $\alpha$, $\beta$. As one example we can consider the physically interesting case $\lambda(t) = a\cos(\omega t)$ which gives $g - \ddot{\lambda}(t) = g + a\omega^2\cos(\omega t)$ with $a$ constant such that $g > |a|\omega^2$; the corresponding solution is given by

$$\theta(t) = \alpha \cos\left(\int \Big(g + a\omega^2\cos(\omega t)\Big)^{\frac{2}{3}}\mathrm{d}t + \beta\right). \tag{25}$$

Note that $\theta(t)$ is in general a quasi-periodic function of time.

*2.5. A class of Hill's equations*

If $Q(t)$ is a periodic function with respect to time then Eq. (9) is known as Hill's equation. One case where Eq. (10) can be solved for time dependent coefficients is when it can be written as Eq. (7) for some function $\omega(t)$. One can see that if this is true, $b(t)$ can be expressed in terms of $a(t)$ and hence $Q(t)$ becomes a function of $a(t)$ only. In this form for the solution $y(t)$ of (9) to be periodic, $a(t)$ must be a periodic function of time with zero mean value. Rewriting equations (10) and $y(t)$ in terms of $a(t)$ and recalling that solutions to (10) where $a(t) = -\dot{\omega}(t)/\omega(t)$ are finite and periodic, we see that all solutions found for the corresponding Hill's equation are in turn bounded and hence stable; so we have

$$\begin{aligned} &\ddot{x}(t) + a(t)\,\dot{x}(t) + \Big(\mathrm{e}^{-2\int a(t)\,\mathrm{d}t}\Big)x = 0, \\ &Q(t) = -\frac{1}{2}\,\dot{a}(t) - \frac{1}{4}\,a^2(t) + \mathrm{e}^{-2\int a(t)\,\mathrm{d}t}, \\ &y(t) = \Big(\mathrm{e}^{\frac{1}{2}\int a(t)\,\mathrm{d}t}\Big)\cos\left(\int \Big(\mathrm{e}^{-\int a(t)\,\mathrm{d}t}\Big)\,\mathrm{d}t\right). \end{aligned} \tag{26}$$

Since $\omega(t) = \mathrm{e}^{-\int a(t)\,\mathrm{d}t}$ the latter solution can be written in the form

$$y(t) = \frac{1}{\sqrt{\omega(t)}}\cos\left(\bar{\omega}t + \Omega(t)\right), \tag{27}$$

where $\bar{\omega}$ is the average of the function $\omega(t)$ and $\Omega(t)$ is a primitive of the zero-average function $\omega(t) - \bar{\omega}$. Note that in general $y(t)$ is a quasi-periodic function, with two frequencies $\bar{\omega}$ and $\omega_0$, with $\omega_0$ being the frequency of $\omega(t)$, and reduces to a purely periodic function only if $\bar{\omega}$ is commensurate with $\omega_0$.

For instance, if one considers Hill's equation

$$\ddot{y}(t) + Q(t)y(t) = 0, \quad Q(t) = \lambda + \Big(\mathrm{e}^{-2\sin^2 t} - \sin^2 t - \sin^4 t\Big),$$

for general values of the parameter $\lambda$ the equation cannot be explicitly solved. In particular, by varying $\lambda$ on the real axis, one passes repeatedly from instability to stability regions [3]. However, for the special value $\lambda = 1$, the solution can be explicitly worked out and it is of the form (27) with $\omega(t) = \mathrm{e}^{-\sin^2 t}$.

More generally one can consider the case in which $\omega(t)$ is quasi-periodic with frequencies $\omega_1, \ldots, \omega_n$. Then the solution $y(t)$ has still the form (26), but with $\Omega(t)$ now a quasi-periodic function with the same frequencies as its derivative $\omega(t)$. Therefore the quasi-periodic Hill's equation (9), with $Q(t)$ as in (13) and $\omega(t)$ quasi-periodic, provides an example where the solution can be completely constructed. This is a remarkable property, since in general the study of the quasi-periodic Hill's equation, especially when the function $Q(t)$ is not close to a constant, requires a very delicate analysis; see for instance [8–11].

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