Chapter 7 Exercises
Solutions

1. *For the transportation problem given by the following tableau, find an initial basic feasible solution by the least-cost method and then find an optimal solution.*

				Supply
	2	1	3	7
	4	5	6	8
Demand	5	6	4	

Initial b.f.s. by the Least Cost method :

				Supply
	1 2	6 1	 3	7
	4 4	 5	4 6	8
Demand	5	6	4	

Assign λ_i and μ_j values and calculate s_{ij} values:

	2	1	4
0	1 2	6 1	−1 3
2	4 4	2 5	4 6

$s_{13} = -1$, so increase x_{13} to ε, which can be as large as 1. x_{11} departs.
x_{21} becomes $4 + \varepsilon$, x_{23} becomes $4 - \varepsilon$.

Assign λ_i and μ_j values and calculate s_{ij} values:

	1	1	3
0	1 2	6 1	1 3
3	5 4	1 5	3 6

This is optimal, as all $s_{ij} \geq 0$.

Minimum cost = 6 + 3 + 20 + 18 = 47.

2. *For the transportation problem given by the following tableau, find an initial basic feasible solution by the North-west corner method and then find an optimal solution.*

					Supply
10	*15*	*10*	*12*	*20*	*8*
5	*10*	*8*	*15*	*10*	*7*
15	*10*	*12*	*12*	*10*	*10*
Demand *5*	*9*	*2*	*4*	*5*	

The supply at Source 3 is now reduced from 10 to 6. There is a penalty of 5 for each unit required but not supplied. Find the new optimal solution.

Initial b.f.s. : $x_{11} = 5, x_{12} = 3, x_{22} = 6, x_{23} = 1, x_{33} = 1,$
$x_{34} = 4, x_{35} = 5.$

Assign λ_i and μ_j values and calculate s_{ij} values:

	10	15	13	13	11
0	5 10	3 15	−3 10	−1 12	9 20
−5	0 5	6 10	1 8	7 15	4 10
−1	6 15	−4 10	1 12	4 12	5 10

$s_{32} = -4$, so increase x_{32} to ε, which can be as large as 1. x_{33} departs. x_{23} becomes $1 + \varepsilon$, x_{22} becomes $6 - \varepsilon$.

	10	15	13	17	15
0	5 10	3 15	−3 10	−5 12	5 20
−5	0 5	5 10	2 8	3 15	0 10
−5	10 15	1 10	4 12	4 12	5 10

$s_{14} = -5$, so increase x_{14} to ε, which can be as large as 3. x_{12} departs. x_{32} becomes $1 + \varepsilon$, x_{34} becomes $4 - \varepsilon$.

	10	10	8	12	10
0	5 10	5 15	2 10	3 12	10 20
0	−5 5	5 10	2 8	3 15	0 10
0	5 15	4 10	4 12	1 12	5 10

$s_{21} = -5$, so increase x_{21} to ε, which can be as large as 1. x_{34} departs. x_{14} becomes $3 + \varepsilon$, x_{11} and x_{22} become $5 - \varepsilon$, x_{32} becomes $4 + \varepsilon$.

	10	15	13	12	15
0	4 10	0 15	−3 10	4 12	5 20
−5	1 5	4 10	2 8	8 15	0 10
−5	10 15	5 10	4 12	5 12	5 10

$s_{13} = -3$, so increase x_{13} to ε, which can be as large as 2. x_{23} departs. x_{21} becomes $1 + \varepsilon$, x_{11} becomes $4 - \varepsilon$.

	10	15	10	12	15
0	2 10	0 15	2 10	4 12	5 20
−5	3 5	4 10	3 8	8 15	0 10
−5	10 15	5 10	7 12	5 12	5 10

All $s_{ij} \geq 0$, so this is optimal. (Solution is not unique, but optimal cost is.)

Minimum cost $= 2(10) + 2(10) + 4(12) + 3(5) + 4(10) + 5(10) + 5(10) = 243$.

Now with supply at source 3 reduced to 6, add dummy row with 'supply' $= 4$.

If we take the previous optimal solution, reduce e.g. x_{35} from 5 to 1, put $x_{45} = 4$, and let $\lambda_4 = -10$, we find that the resulting tableau is still optimal. The cost is now 223.

3. *Three refineries with maximum daily capacities of 6, 5, and 8 million gallons of oil supply three distribution areas with daily demands of 4, 8 and 7 million gallons. Oil is transported to the three distribution areas through a network of pipes. The transportation cost is 1 pence per 100 gallons per mile. The mileage table below shows that refinery 1 is not connected to distribution area 3. Formulate the problem as a transportation model and solve it. [Hint: Let the transportation cost for the non-connected route be equal to some large value M say and then proceed as normal.]*

Refinery	Distribution Area 1	2	3
1	120	180	—
2	300	100	80
3	200	250	120

The cost of transporting one million gallons per mile is 10,000 pence or £100. Thus, when working in units of millions of gallons, the mileage figures should be multiplied by 100 to give the cost per mile.

The optimal cost $= 2,430 \times 10 =$ £243,000 when e.g. $x_{11} = 4,\ x_{12} = 2,\ x_{22} = 5,\ x_{32} = 1,\ x_{33} = 7$, all in millions of gallons. This is obtained directly as the initial solution from the NW Corner method, or with one iteration from the least-cost method.

	120	180	50	
0	4 120	2 180	 M	6
-80	260 300	5 100	50 80	5
70	10 200	1 250	7 120	8
	4	8	7	

4. *In problem 3, suppose additionally that the capacity of refinery 3 is reduced to 6 million gallons. Also, distribution area 1 must receive all its demand, and any shortage at areas 2 and 3 will result in a penalty of 5 pence per gallon. Formulate the problem as a transportation model and solve it.*

We introduce a dummy refinery with a capacity of 2 million gallons to balance the problem and a penalty cost of £5 per 100 gallons. We also assign 4 to x_{11} and, necessarily, 2 to x_{12} and then apply the least cost method to the remaining cells. The cost is now £348,000 and the tableau is

	120	180	50	
0	4 120	2 180	 M	6
30	150 300	ϵ -110 100	$5-\epsilon$ 80	5
70	10 200	$4-\epsilon$ 250	$2+\epsilon$ 120	6
48820	300 50000	2 50000	230 50000	2
	4	8	7	

The tableau is not optimal as the shadow cost for x_{33} is negative. We find that $\epsilon = 4$, the total cost is £304,000 and the optimal tableau is

	120	180	50	
0	4 120	2 180	M	6
-80	260 300	4 100	1 80	5
-40	40 200	110 250	2 120	6
48820	340 50000	2 50000	230 50000	2
	4	8	7	

5. *In problem 3, suppose the daily demand at area 3 drops to 4 million gallons. Any surplus production at refineries 1 and 2 must be diverted to other distribution areas by tanker. The resulting average transportation costs per 100 gallons are £1.50 from refinery 1 and £2.20 from refinery 2. Refinery 3 can divert its surplus oil to other chemical processes within the plant. Formulate the problem as a transportation model and solve it.*

We introduce a dummy area with a demand of 3 with the given transportation costs, setting $c_{33} = 0$. We proceed as before, we assign 4 to x_{11} and, necessarily, 2 to x_{12} and then apply the least cost method to the remaining cells.

	120	180	50	-70	
0	4 120	2 180	M	220 150	6
-80	260 300	$1+\epsilon$ 100	$4-\epsilon$ 80	370 220	5
70	10 200	$5-\epsilon$ 250	ϵ -110 120	3 0	8
	4	8	4	3	

The cost is £251,000 and the solution is not optimal since $s_{33} < 0$. We have $\epsilon = 4$ and iterat to get the optimal tableau with cosrt £207,000.

	120	180	50	-70	
0	4 120	2 180	M	220 150	6
-80	260 300	5 100	50 80	370 220	5
70	10 200	1 250	4 120	3 0	8
	4	8	4	3	

6. *Five warehouses are supplied by four factories. The supply available from each factory, the demand at each warehouse and the cost per unit of transporting goods from the factories to the warehouses are summarised in the following table:*

	W_1	W_2	W_3	W_4	W_5	*Supply*
Factory 1	*13*	*9*	*15*	*10*	*12*	*40*
Factory 2	*11*	*10*	*12*	*12*	*9*	*10*
Factory 3	*12*	*9*	*11*	*12*	*9*	*20*
Factory 4	*13*	*12*	*13*	*12*	*10*	*10*
Demand	*12*	*15*	*20*	*15*	*18*	

(a) *Use the North-West Corner method to find an initial basic feasible solution of this problem. [Do NOT use the Least-Cost method].*

The initial basic feasible solution is as follows, the cost is 922:

	W_1	W_2	W_3	W_4	W_5	Supply
F_1	12 / 13	15 / 9	13 / 15	10	12	40
F_2	11	10	7 / 12	3 / 12	9	10
F_3	12	9	11	12 / 12	8 / 9	20
F_4	13	12	13	12	10 / 10	10
Demand	12	15	20	15	18	

(b) *Find the optimal solution of this problem, i.e. the solution that minimises the transportation costs, clearly showing and explaining your working.*

We suppose that the cost of transporting 1 unit from factory i to warehouse j, c_{ij} is made up of a dispatch cost λ_i and a reception cost μ_j such that

$$c_{ij} = \lambda_i + \mu_j.$$

Since there are $n + m$ of the λ and μ variables but only $n + m - 1$ basic variables we can arbitrarily assign a value to one of the λ_i. We put $\lambda_1 = 0$ and compute the other values for λ_i and μ_j.

The shadow cost is the difference for each cell between the value of c_{ij}, computed in this manner and the transportation cost. All of these shadow prices must be positive for an optimal solution.

The largest negative shadow price is that of cell (14). Assigning ϵ units to this cell and making the concomitant adjustments to the other cells shows that $\epsilon \leq 3$.

	13	9	15	15	12	Supply
0	12 13	15 9	$13-\epsilon$ 15	ϵ -5 10	0 12	40
-3	1 11	4 10	$7+\epsilon$ 12	$3-\epsilon$ 12	0 9	10
-3	2 12	3 9	-1 11	12 12	8 9	20
-2	2 13	5 12	0 13	-1 12	10 10	10
Demand	12	15	20	15	18	

The departing variable is x_{24} The new tableau is as follows, with the λ_i and μ_j values shown. Computing the s_{ij} values we find that the solution is not optimal as not all $s_{ij} \geq 0$. The current cost is 907. We see that the entering variable is now x_{33} and the amount it can be increased is 10, while the departing variable is x_{13}.

	13	9	15	10	11	Supply
0	12 13	15 9	$10-\epsilon$ 15	$3+\epsilon$ 10	5 12	40
-3	1 11	4 10	10 12	5 12	5 9	10
2	-3 12	-2 9	ϵ -6 11	$12-\epsilon$ 12	8 9	20
3	-3 13	0 12	-5 13	-1 12	10 10	10
Demand	12	15	20	15	18	

Making the necessary changes we obtain the next tableau with the λ and μ values shown. The current cost is 847. Not all the $s_{ij} \geq 0$, so the solution is not optimal. The entering variable is now x_{21} and the amount it can be increased is 2, while the departing variable is x_{34}.

	13	9	9	10	7	Supply
0	$12-\epsilon$ 13	15 9	6 15	$13+\epsilon$ 10	5 12	40
3	ϵ -5 11	-2 10	$10-\epsilon$ 12	-1 12	-1 9	10
2	-4 12	2 9	$10+\epsilon$ 11	$2-\epsilon$ 12	8 9	20
3	-3 13	6 12	7 13	-1 12	10 10	10
Demand	12	15	20	15	18	

Making the necessary adjustments we obtain the next tableau, total cost is 837. Not all the s_{ij} are non-negative so we do not have the optimal solution. The entering variable is x_{25} which can be increased by 8. We see that we have a solution where both x_{35} and x_{23} are zero. Only one of them can be the departing variable because we must always have $5 + 4 - 1 = 8$ basic variables. In this case one of our basic variables will take the value zero.

	13	9	14	10	12	Supply
0	10 13	15 9	1 15	15 10	0 12	40
-2	2 11	3 10	$8 - \epsilon$ 12	4 12	ϵ -1 9	10
-3	2 12	3 9	$12 + \epsilon$ 11	5 12	$8 - \epsilon$ 9	20
-2	2 13	5 12	1 13	4 12	10 10	10
Demand	12	15	20	15	18	

With these adjustments the tableau is as follows, the cost is now 829. In order to assign λ_i and μ_j values we need to set one of x_{35} or x_{23} as the zero basic variable. We choose x_{23} arbitrarily and obtain a set of non-negative s_{ij} values, thus we have found the optimum solution. (We obtain a different, but still non-negative set of s_{ij} values if we choose x_{35}.)

	13	9	14	10	11	Supply
0	10 13	15 9	1 15	15 10	0 12	40
-2	2 11	3 10	0 12	2 12	8 9	10
-3	2 12	3 9	20 11	5 12	1 9	20
-2	2 13	5 12	1 13	4 12	10 10	10
Demand	12	15	20	15	18	

The solution is thus $x_{11} = 10, x_{12} = 15, x_{14} = 15, x_{21} = 2, (x_{23} = 0), x_{25} = 9, x_{33} = 20, x_{45} = 10$ and cost $= 829$.

7. *For the transportation problem given by the following tableau, find an initial basic feasible solution by the North-West corner method and then find an optimal solution.*

				Supply
	9	15	12	*10*
	6	8	13	*23*
	9	3	11	*27*
Demand	*21*	*14*	*25*	

Using the North-West corner method gives the following initial basic feasible solution:

				Supply
	10 9	15	12	10
	11 6	12 8	13	23
	9	2 3	25 11	27
Demand	21	14	25	

Adding the λ_i's, the μ_j's and the s_{ij}'s gives

	9	11	19
0	10 9	4 15	-7 12
-3	11 6	12 8	-3 13
-8	8 9	2 3	25 11

The entering variable is now x_{13} and we take $\varepsilon = 10$. The departing variable is x_{11}. We then obtain the new solution:

	2	4	12
0	7 9	11 15	10 12
4	21 6	2 8	-3 13
-1	8 9	12 3	15 11

Entering variable: x_{23} with $\varepsilon = 2$.
Departing variable: x_{22}. [2]

	5	4	12
0	4 9	11 15	10 12
1	21 6	3 8	2 13
-1	5 9	14 3	13 11

This is now the optimal solution as all the $s_{ij} \geq 0$. The minimum cost is then

$$10 \times 12 + 21 \times 6 + 2 \times 13 + 14 \times 3 + 13 \times 11 = 457$$