Chapter 2 Exercises

Solutions

Question 1

Find the optimal solution for the two container problem with the following assumptions:

- *Container A requires M_1 for 4 minutes and M_2 for 4 minutes.*
- *Container B requires M_1 for 5 minutes and M_2 for 4 minutes.*
- *The net profit for container A is £29 and the net profit for container B is £45.*

Solution

Let x_1, x_2 be the numbers of containers of type A and type B respectively manufactured per hour.
The objective function is profit per hour;

$$P = 29x_1 + 45x_2$$

The constraints are

$$4x_1 + 5x_2 \leq 60 \quad \text{constraint 1}$$

$$4x_1 + 4x_2 \leq 60 \quad \text{constraint 2}$$

with non-negativity conditions $x_1 \geq 0, x_2 \geq 0$.

The feasible region is as shown in Figure 1. The optimal solution is $(0, 12)$ with $P_{max} = 45 \times 12 = 540$.

1. *If the company were able to increase the availability of time on either machine M_1 or machine M_2 which should they choose and why?*

   Machine 2 constraint is redundant, so the company should not increase time on this machine. The constraint on machine 1 is active, so increasing the availability of this machine would increase profit.

2. *If they were to chose to increase the availability of time on machine M_1, how many extra minutes per hour would maximise profit?*

   The time available on machine 1 should be ioncreased until the line representing the constraint on machine 1 intersects the point $(0, 15)$, that is the point at which the constraint on machine 2 will become active. This corrsponds to the equation $4x_1 + 5x_2 = 75$ and indicates an extra 15 minutes of avalibailty on machine 1.

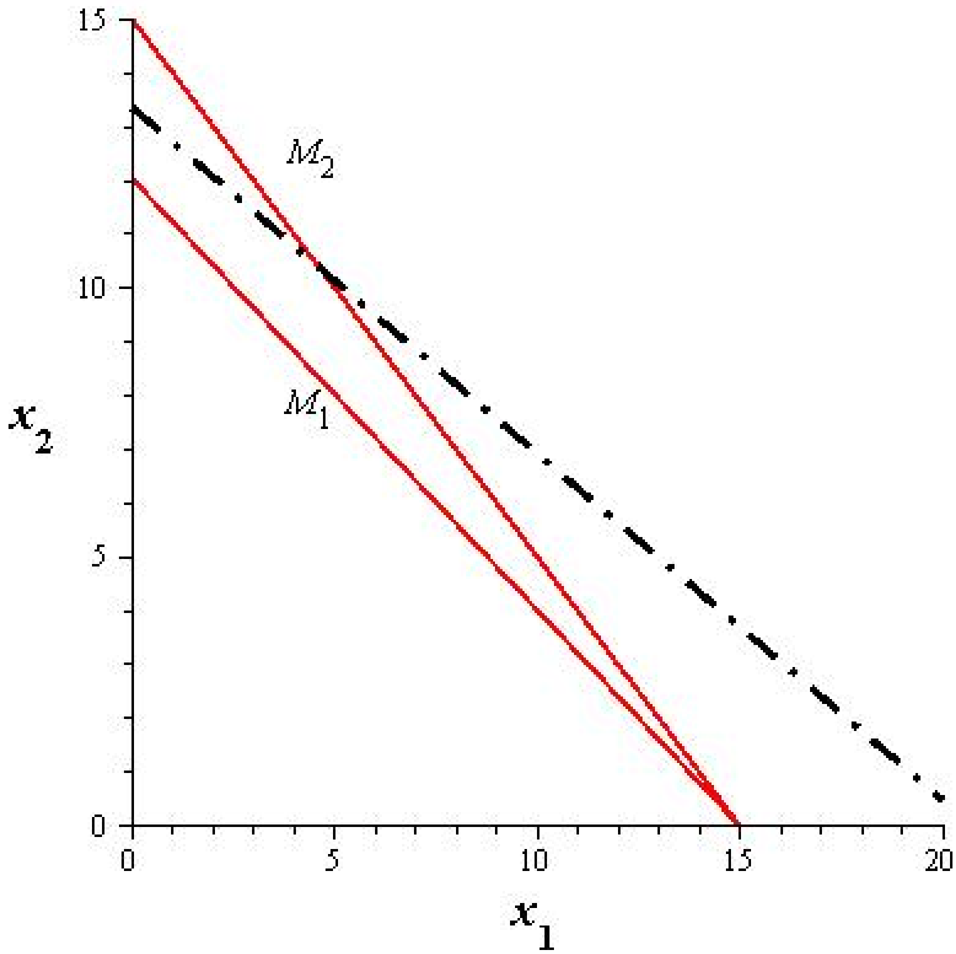


Figure 1: feasible region for container problem

Question 2

Find the optimal solutions for the rose growing problem with the same constraints as in the notes but with the following objective functions:

1. $f(r, w) = 3r + 2w$

   We can find the answer graphically (as was shown in the tutorial) or more analytically, (using the fact that the feasible region must be convex) as follows.

   We calculate the slope of the line representing the land constraint as follows, (w is the variable we represent on the vertical coordinate)

$$5r + 4w = 6000 \Rightarrow w = \frac{6100}{4} - \frac{5}{4}r \Rightarrow \frac{dw}{dr} = -\frac{5}{4}$$

   In a simialr manner we find that of the finance constraint is -4 and that of labour $-\frac{1}{5}$. The slope of the objective function is $-\frac{3}{2}$. This lies between the slopes of finance and land.

$$-4 < -\frac{3}{2} < -\frac{5}{4}.$$

The optimum solution is thus at the intersection of the finance and land lines.

We solve

$$5r + 4w = 6100, \quad 8r + 2w = 800 \Rightarrow r = 900, \quad w = 400, \quad f(900, 400) = 3500.$$

2. $f(r, w) = r + w$

   The slope of the objective function is -1 and this slope lies between those of land and labour

$$-\frac{5}{4} < 1 < -\frac{1}{5}$$

   The optimum solution is thus at the intersection of the labour and land lines.

   We solve

$$r + 4w = 6100, \quad r + 5w = 5000 \Rightarrow r = 500, \quad w = 900, \quad f(500, 900) = 1400$$

3. $f(r, w) = r + 6w$

   The slope of the objective function is $-\frac{1}{6}$ This slope is greater than all of the slopes of the constraints and in particular, greater than the labour line.

$$-\frac{1}{6} > -\frac{1}{5}$$

   and the solution is now at the intersection of the labour line with the w axis, $r = 0, w = 1000, f(0, 1000) = 6000$

4. $f(r, w) = 2.5r + 2w$

   The slope of this objective function is the same as that of the land constraint, $-\frac{5}{4}$. Thus any point on the area line in the feasible region will be optimal.

   e.g

$$f(500, 900) = 3050, \quad f(900, 400) = 3500$$

Question 3

A boat building firm makes two types of boat, a family row-boat and a sports canoe. The boats are moulded from aluminium using a machine press and then are finished by hand. The row-boat requires 25kg of aluminium, 6 minutes of machine time and 3 hours of hand labour.

A canoe requires 15kg of aluminium, 5 minutes of machine time and 5 hours of hand labour. Over the next month, the firm can commit up to 1000kg of aluminium, 5 hours of machine time and 200 hours of hand labour. They make a profit of £25 on each row-boat sold and £30 on each canoe. Assuming they can sell all the boats they make, determine how many of each type they should make in order to maximise their profit.

Solution

The decision variables are

$$r = \text{number of rowboats manufactured per month}$$
$$c = \text{number of canoes manufactured per month}$$

The constraints are

$$25r + 15c \leq 1000 \quad \text{Aluminium, units are } \textbf{kgs}$$

$$6r + 5c \leq 300 \quad \text{machine time, units are } \textbf{minutes}$$

$$3r + 5c \leq 200 \quad \text{hand labour, units are } \textbf{hours}.$$

The objective function is profit in £ per month which is

$$f(r, c) = 25r + 30c.$$

In order to find the feasible region we draw the graph as in figure 2 below.

We note that the machine time constraint is redundant, no part of the corresponding line appears in the feasible region.

The objective line shown is $25r + 30c = 1500$ and it is clear that the point at which the contour just touches the feasible region is at the intersection of the Aluminium and hand labour constraints, this is thus the optimum solution.

The coordinates of this optimum solution could be read from an accurate graph but precise co-ordinates of this point are found by solving simultaneously

$$25r + 15c = 1000 \quad \text{and} \quad 3r + 5c = 200.$$

Subtracting three times the second equation from the first we have

$$16r = 400 \quad \text{thus} \quad r = 25 \quad \text{and} \quad c = 25$$

The value of the objective function is therefore $f(25, 25) = 1375$.

We could also have found the optimum solution by noting that the slope of the objective line (-5/6) lies between the slopes of the Aluminium line (-5/3) and the hand labour line (-3/5) and that the intersection of these two lines will give the optimum solution.

Thus the company should manufacture 25 row boats and 25 canoes each month and this will yield a monthly profit of £1375.

We note that the slope of the objective line is

$$-\frac{\text{profit per row boat}}{\text{profit per canoe}}$$

so that increasing the profit per row boat will increase the slope of the objective line (making it more negative) while increasing the profit on canoes will decrease the slope of the objective line (making it less negative - or more positive). As the slope is decreased to the point where it is less steep than the slope of the hand labour line, the optimum point will shift to $(0, 40)$ and the company should make only canoes to maximise its profit.

Let the profit at which the company should begin to make only canoes be p per canoe. Then the objective function is $25r + pc$, its slope $-\frac{25}{p}$ and we require that

$$-\frac{25}{p} \geq -\frac{3}{5} \quad \text{(satisfy yourself about the direction of the inequality)}$$

$$3p \geq 125 \quad \text{so} \quad p \geq 41\frac{2}{3}.$$

Thus once the profit per canoe exceeds £41.67 the company should manufacture only canoes.

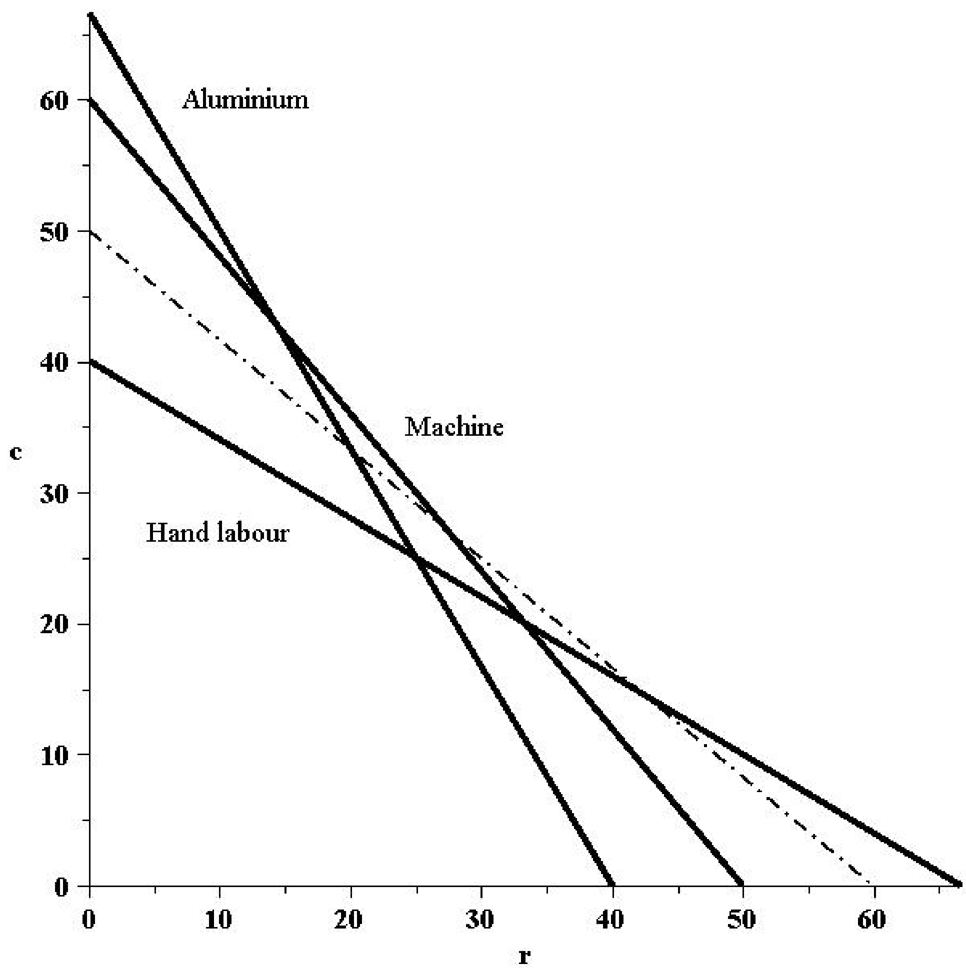


Figure 2: feasible region for rowboat and canoe problem

Question 4

Put the following problems into standard form but **do not try to solve them**.

1.

$$\textbf{\textit{Minimise}} \quad z = 3x_1 - 2x_2$$

$$\textbf{\textit{subject to}} \quad \left\{ \begin{array}{rcrcr} 2x_1 & + & x_2 & \leq & 10 \\ -x_1 & + & 4x_2 & \leq & -1 \end{array} \right.$$

$$\textbf{\textit{and}} \quad x_1, x_2 \geq 0.$$

Solution

Maximise $z = -3x_1 + 2x_2$ (Finding the value of x which minimises $f(x)$ is precisely equivalent to finding the value of x which maximises $-f(x)$.)

Subject to

$2x_1 + x_2 + x_3 = 10$ add the slack variable x_3 to turn the inequality into an equation

For the second constraint, first add a slack variable in the usual way, $-x_1+4x_2+x_4 = -1$, and then multiply both sides by -1 to ensure there is a non-negative constant on the right hand side

$$x_1 - 4x_2 - x_4 = 1.$$

The non-negativity constraints are

$$x_i \geq 0 \quad \text{for} \quad i = 1, ..., 4.$$

2.

$$\textbf{\textit{Minimise}} \qquad z = 2x_1 + 3x_2$$

$$\textbf{\textit{subject to}} \quad \begin{cases} -2x_1 & + & 3x_2 & \geq & -5 \\ 7x_1 & - & 4x_2 & \geq & 6 \end{cases}$$

$$\textbf{\textit{and}} \qquad x_1 \ \textit{unrestricted}, \ x_2 \geq 0.$$

Solution

Maximise $z = -2x_1 - 3x_2$. (process as in (1) above)

Subject to

$$-2x_1 + 3x_2 - x_3 = -5$$

(subtracting a slack variable in the first constraint), which is equivalent to $2x_1 - 3x_2 + x_3 = 5$ (multiplying by -1 to get a non-negative right hand side), and

$$7x_1 - 4x_2 - x_4 = 6$$

(subtracting a slack variable in the second constraint).

x_1 is unrestricted so replace it by $x_1 = x' - x''$ where $x' \geq 0, x'' \geq 0$. Now substitute for x_1 in the objective function and in the constraint equations we have just obtained.

Thus we have restated the problem in standard form as follows.

Maximise

$$z = -2x' + 2x'' - 3x_2$$

Subject to

$$x' - x'' + x_2 = 10$$

$$2x' - 2x'' - 3x_2 - x_3 = 5$$

$$7x' - 7x'' - 4x_2 + x_4 = 6$$

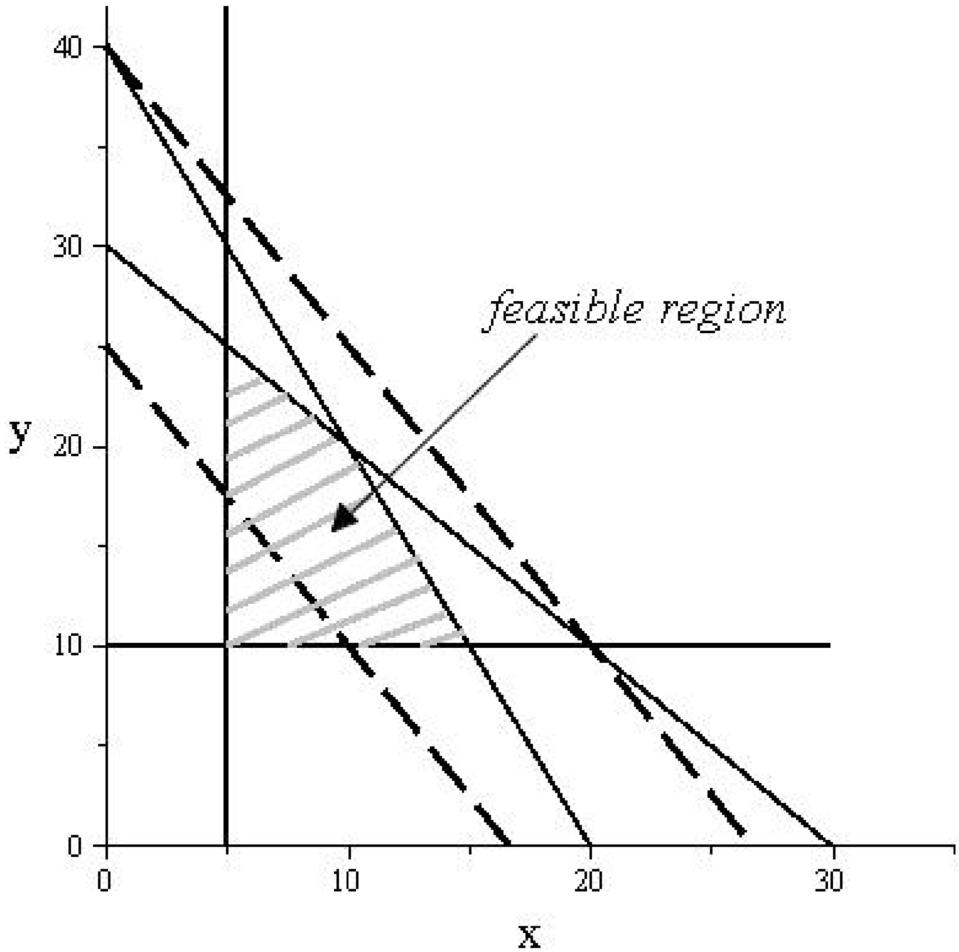
$$x', x'', x_2, x_3, x_4 \geq 0$$

Question 5

A builder has a plot of land available on which he can build either luxury of standard houses. He decides to build at least 5 luxury and 10 standard houses. Planning restrictions limit him to no more than 30 houses altogether. A luxury house requires $300m^2$ *of land and a standard house* $150m^2$*. The plot is* $6000m^2$*. Profit is* £$12,000$ *per luxury house and* £8000 *per standard house. How many of each type should he build to maximise his profit.*

Solution

- We take x for the number of luxury house and y for the number of standard houses.
- The constraints are as follows
  - $x > ge5$
  - $y \geq 10$
  - the planning limit is $x + y \leq 30$
  - land availability $300x + 150y \leq 6000$ or $2x + y \leq 40$
- The objective function is the maximisation of profit on the entire project so $Z = 12,000x + 8,000y$ The feasible area is found from the following graph



The graph shows two contours $Z = 200,000$ and $Z = 320,000$ from which is is clear that the optimum solution is $x = 10, y = 20$ for a maximum profit of $£280,000$

- The builder should construct 10 luxury houses and 20 standard houses and thus maximise his profit at £280,000.