Answers to Exercises in MS100 Notes (Part B)

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1. (a) $\int \sinh(3x-4)dx = \frac{1}{3}\cosh(3x-4)+c$

   (b) $5\int_0^1 \frac{1}{\sqrt{2^2-x^2}}dx = 5\left[\arcsin\frac{x}{2}\right]_0^1 = 5\arcsin\frac{1}{2} = \frac{5\pi}{6}$

   (c) $\int 3^x + \frac{\sin x}{\cos x+1}dx = \frac{3^x}{\ln 3} - \ln(\cos x+1)+c$, using $\int \frac{f'(x)}{f(x)} = \ln|f(x)|+c$ for the second term.

   (d) $\int \frac{1}{(x+1)^2+1}\ dx = \arctan(x+1)+c$

   (e) $\int_3^4 \frac{1}{(2x-5)^2}\ dx = \int_3^4 (2x-5)^{-2}\ dx = \left[-\frac{1}{2(2x-5)}\right]_3^4 = \frac{1}{3}.$

2. (a) $\int \frac{1}{\sqrt{x^2-4x+3}}\ dx = \int \frac{1}{(x-2)^2-1}\ dx = \text{arcosh}(x-2)+c.\ (x \geq 3)$

   (b) Split up the integral as $\int \frac{x}{\sqrt{x^2-1}}\ dx - \int \frac{2}{\sqrt{x^2-1}}\ dx.$
   For first term, let $u = x^2-1, du = 2x\ dx$, to get
   $\int \frac{1}{2}u^{-1/2}\ du = u^{1/2}+c = (x^2-1)^{1/2}+c.$
   Second term integrates to 2 arcosh x, so we get $\sqrt{x^2-1} - 2\ \text{arcosh}\ x + c.$

   (c) Let $u = 1-x^2$ so $du = -2x\ dx.$ $\int -\frac{1}{2}u^{-1/2}\ du = -u^{1/2}+c = c-\sqrt{1-x^2}.$
   OR let $x = \sin\theta, dx = \cos\theta\ d\theta$, to get $\int \sin\theta\ d\theta = -\cos\theta + c = c-\sqrt{1-x^2}.$

   (d) Let $u^2 = x^2-x+1$, so $2u\ du = (2x-1)\ dx.$
   Integral is $\int \frac{u}{u}\ du + \frac{3}{2\sqrt{(x-1/2)^2+3/4}}\ dx$
   $= u + \frac{3}{2}\ \text{arsinh}\ \frac{x-1/2}{\sqrt{3}/2} + c = \sqrt{x^2-x+1} + \frac{3}{2}\ \text{arsinh}\ \frac{2x-1}{\sqrt{3}} + c.$

   (e) Let $x = 3\sec u$ so $\sqrt{x^2-9} = 3\tan u,\ dx = 3\sec u\tan u du$
   $\int \frac{\tan u}{\sec u}\ 3\sec u\tan u\ du = 3\int \tan^2 u\ du = 3\int (\sec^2 u - 1)\ du = 3(\tan u - u)+c$
   $= 3\left(\sqrt{\frac{x^2}{9}-1} - \text{arcsec}\ \frac{x}{3}\right) + c = \sqrt{x^2-9} - 3\arccos\frac{3}{x} + c.$

   (f) Integral is $\int \sqrt{1-(x-1)^2}\ dx$. Let $x-1 = \sin t$, so $dx = \cos t\ dt$.
   We get $\int \cos^2 t\ dt = \int \frac{1}{2}(\cos 2t + 1)\ dt = \frac{1}{4}\sin 2t + \frac{1}{2}t + c$
   $= \frac{1}{2}(\sin t\cos t + t) + c = \frac{1}{2}((x-1)\sqrt{2x-x^2} + \arcsin(x-1)) + c.$

3. (a) Integral is $\int \frac{1}{\tan x + 1}\,dx$. Let $u = \tan x$, so $du = \sec^2 x\,dx = (1+u^2)\,dx$.

We get $\int \frac{1}{1+u}\frac{du}{1+u^2} = \int \frac{1}{(1+u)(1+u^2)}\,du$

$= \frac{1}{2}\int (\frac{1}{1+u} + \frac{1-u}{1+u^2})\,du$ (by Partial Fractions)

$= \frac{1}{2}[\ln(1+u) + \arctan u - \frac{1}{2}\ln(1+u^2)] + c$

$= \frac{1}{2}\ln|1+\tan x| - \frac{1}{4}\ln(1+\tan^2 x) + \frac{x}{2} + c.$

(b) Let $t = \tan\frac{x}{2}$, so we have $\int \frac{1}{2-2t/(1+t^2)}\frac{2}{1+t^2}\,dt$

$= \int \frac{1}{1+t^2-t}\,dt = \int \frac{1}{(t-1/2)^2+3/4}\,dt = \arctan\frac{t-1/2}{\sqrt{3}/2} + c$

$= \frac{2}{\sqrt{3}}\arctan\frac{2t-1}{\sqrt{3}} + c = \frac{2}{\sqrt{3}}\arctan\frac{2\tan(x/2)-1}{\sqrt{3}} + c$

(c) Let $x = \sin t$, so $dx = \cos t\,dt$. Integral becomes $\int_{-\pi/2}^{\pi/2}\cos^2 t\,dt$

$= 2\int_0^{\pi/2}\frac{1}{2}(\cos 2t + 1)dt = [-\frac{1}{2}\sin 2t + t]_0^{\pi/2} = \frac{\pi}{2}.$

4. (a) $\frac{x^2}{x^2-4x+3} = 1 + \frac{4x-3}{x^2-4x+3}.$

Let $\frac{4x-3}{x^2-4x+3} \equiv \frac{A}{x-1} + \frac{B}{x-3}$. We get $A = -\frac{1}{2}$, $B = \frac{9}{2}$.

Then the integral is $x - \frac{1}{2}\ln|x-1| + \frac{9}{2}\ln|x-3| + c.$

(b) $\frac{x+1}{x(x+5)^2} = \frac{1}{25}\left(\frac{1}{x} - \frac{1}{x+5} + \frac{20}{(x+5)^2}\right)$

Thus the integral is $\frac{1}{25}\left(\ln|x| - \ln|x+5| - \frac{20}{x+5}\right) + c.$

(c) $\frac{x^3}{x^3+x^2-4x-4} = 1 - \frac{x^2-4x-4}{(x+1)(x+2)(x-2)}.$

In partial fractions, this is $1 + \frac{1}{3(x+1)} + \frac{2}{3(x-2)} - \frac{2}{x+2}.$

Integral is $x + \frac{1}{3}\ln|x+1| + \frac{2}{3}\ln|x-2| - 2\ln|x+2| + c.$

(d) $\frac{x^3+4x^2-x+3}{x^3-2x^2+x-2} = 1 + \frac{6x^2-2x+5}{(x-2)(x^2+1)} = 1 + \frac{5}{x-2} + \frac{x}{x^2+1}.$

Integral is $x + 5\ln|x-2| + \frac{1}{2}\ln(x^2+1) + c,$

using $\int \frac{f'(x)}{f(x)} = \ln|f(x)| + c$ for the last term.

5. Dividing top and bottom by $\cos^2 x$, the integral is $\displaystyle\int_0^{\pi/2} \frac{\sec^2 x}{a^2 + b^2 \tan^2 x}\, dx$.

Let $u = \tan x$, so we get $\displaystyle\int_0^{\infty} \frac{1}{a^2 + b^2 u^2}\, du, = \lim_{t\to\infty} \int_0^{t} \frac{1}{a^2 + b^2 u^2}\, du,$

$$= \lim_{t\to\infty} \left[\frac{1}{ab} \arctan \frac{bu}{a}\right]_0^t = \frac{1}{ab} \lim_{t\to\infty} \arctan \frac{bt}{a} = \frac{1}{ab}\frac{\pi}{2} = \frac{\pi}{2ab}, \text{ as required.}$$

6. Mean value $= \displaystyle\frac{1}{6}\int_0^6 \frac{1}{(x+2)(x+3)}\, dx = \frac{1}{6}\int_0^6 \frac{1}{x+2} - \frac{1}{x+3}\, dx$

$$= \frac{1}{6}[\ln(x+2) - \ln(x+3)]_0^6 = \frac{1}{6}(\ln 8 - \ln 9 - \ln 2 + \ln 3) = \frac{1}{6}\ln\frac{4}{3}.$$

The curve decreases from (0, 1/6) to (6, 1/72). The mean value of about 0.05 is believable as the height of a rectangle, of length 6, whose area equals that under the curve.

7. $\displaystyle\frac{dy}{dx} = \frac{6\cosh t \sinh t}{6\sinh^2 t \cosh t} = \operatorname{cosech} t \qquad \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} = \coth t$

Arc length $= \displaystyle\int_0^{\ln 3} \coth t.6\sinh^2 t \cosh t\ dt = \left[2\cosh^3 t\right]_0^{\ln 3}$

$$= 2\left[\left(\frac{3 + 1/3}{2}\right)^3 + 1\right] = \frac{196}{27}.$$

8. $\displaystyle\frac{dy}{dx} = \sin x \qquad \left(1 + \left(\frac{dy}{dx}\right)^2\right)^{1/2} = \sqrt{1 + \sin^2 x}$

Area $= \displaystyle 2\pi \int_0^{\pi/2} \cos x \sqrt{1 + \sin^2 x}\, dx$ Let $u = \sin x$, $du = \cos x\ dx$

Limits become $u = 0, 1$. Area $= \displaystyle 2\pi \int_0^1 \sqrt{1 + u^2}\, du$

Let $u = \sinh t$, so $du = \cosh t\ dt$. Area $= \displaystyle 2\pi \int_0^{\text{arsinh}1} \cosh^2 t\ dt$

$$= \pi \int_0^{ar\sinh 1} \cosh 2t + 1\ dt = \pi\left[\frac{\sinh 2t}{2} + t\right]_0^{\text{arsinh}1} = \pi(\text{arsinh } 1 + \sqrt{2})\ .$$

9. Let $u = x^n$, $dv = \cosh x$ $\quad$ $dxdu = nx^{n-1}\ dx$, $v = \sinh x$

$I_n = [x^n \sinh x]_0^1 - n\int_0^1 x^{n-1} \sinh x\ dx$

Let $u = x^{n-1}$, $dv = \sinh x\ dx$ $\quad$ $du = (n-1)x^{n-2}\ dx$, $v = \cosh x$

$I_n = [x^n \sinh x - nx^{n-1} \cosh x]_0^1 - n(n-1)\int_0^1 x^{n-2} \cosh x\ dx$

$I_n = \sinh 1 - n \cosh 1 + n(n-1)I_{n-2}$

$I_0 = \int_0^1 \cosh x\ dx = \sinh 1$

$I_2 = \sinh 1 - 2 \cosh 1 + 2 \sinh 1 = 3 \sinh 1 - 2 \cosh 1$

$I_4 = \sinh 1 - 4 \cosh 1 + 12(3 \sinh 1 - 2 \cosh 1) = 37 \sinh 1 - 28 \cosh 1$

10. (a) $\displaystyle\int_1^{\infty} \frac{1}{x^3}\ dx = \lim_{b\to\infty}\left[\frac{-1}{2x^2}\right]_1^b = \lim_{b\to\infty}\left[\frac{-1}{2b^2} - \frac{-1}{2}\right] = \frac{1}{2}.$

(b) $\displaystyle\int_0^2 x^{-1/3}\,\mathrm{d}x = \lim_{a\to 0+}\left[\frac{3}{2}x^{2/3}\right]_a^2 = \lim_{a\to 0+}\left[\frac{3}{2}2^{2/3} - \frac{3}{2}a^{2/3}\right] = \frac{3}{2^{1/3}}.$

(c) $\displaystyle\int_{-\infty}^{-2} \frac{-1}{x^2+4}\,\mathrm{d}x = \lim_{a\to -\infty}\left[-\frac{1}{2}\arctan\frac{x}{2}\right]_a^{-2}$

$\displaystyle= \lim_{a\to -\infty}\left[-\frac{1}{2}\arctan(-1) + \frac{1}{2}\arctan\frac{a}{2}\right] = \frac{1}{2}\left(\frac{\pi}{4} - \frac{\pi}{2}\right) = -\frac{\pi}{8}.$

(d) $\displaystyle\int_0^{\pi/2} \tan x\,\mathrm{d}x = \lim_{b\to\pi/2}\left[\ln\sec x\right]_0^b = \lim_{b\to\pi/2}\left[\ln\sec b - 0\right] = \infty,$

so integral does not exist.

(e) $\displaystyle\int_3^4 \frac{1}{\sqrt{x^2-4x+3}}\,\mathrm{d}x = \lim_{a\to 3}\int_a^4 \frac{1}{\sqrt{(x-2)^2-1}}\,\mathrm{d}x = \left[\operatorname{arcosh}\,(x-2)\right]_3^4$

$= \operatorname{arcosh}\,2 - \operatorname{arcosh}\,1 = \operatorname{arcosh}\,2 = \ln(2+\sqrt{3}).$