Answers to Exercises in MS100 Notes (Part B)

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1. $f(0) = 0$. $f'(x) = \sec^2 x$ so $f'(0) = 1$.
   $f''(x) = 2\sec^2 x \tan x$ so $f''(0) = 0$. $f'''(x) = 4\sec^2 x \tan^2 x + 2\sec^4 x$ so $f'''(0) = 2$.
   Thus $\tan x = x + \frac{1}{3}x^3 + \ldots$.

2. $(1 + x^2)^{-1} = 1 - x^2 + x^4 - \ldots$, valid for $|x| < 1$.
   Integrating, we get $\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \ldots$, also valid for $|x| < 1$.

3. $\cos x \sinh x = \left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots\right)\left(x + \frac{x^3}{6} + \frac{x^5}{120} - \ldots\right) = x - \frac{1}{3}x^3 - \frac{1}{30}x^5 + \ldots$.

4. $r = |z| = \sqrt{(6+2)} = \sqrt{8}$ and $\theta = \arg(z) = \arctan(1/\sqrt{3}) = \pi/6$.
   $z = \sqrt{8}\ e^{i\pi/6}$. We can also write z as $\sqrt{8}\ e^{13i\pi/6}$ and $\sqrt{8}e^{25i\pi/6}$.
   $z^{1/3} = \sqrt{2}\ e^{i\pi/18}, \sqrt{2}\ e^{13i\pi/18}, \sqrt{2}\ e^{25i\pi/18}$.

5. Square roots of 16 are 4, –4, so fourth roots of 16 are 2, –2, 2i, –2i.

6. Fifth roots of 1 are 1, $e^{2\pi i/5}$, $e^{4\pi i/5}$, $e^{6\pi i/5}$, $e^{8\pi i/5}$.
   If $u^5 = 1$ and $v = u + \frac{1}{u}$,
   then $v^2 + v - 1 = u^2 + 2 + \frac{1}{u^2} + u + \frac{1}{u} - 1 = \frac{u^4 + u^3 + u^2 + u + 1}{u^2}$.
   Now $(u^5 - 1) = (u-1)(u^4 + u^3 + u^2 + u + 1)$, so if $u \neq 1$ then $u^4 + u^3 + u^2 + u + 1 = 0$.
   Hence $v^2 + v - 1 = 0$.
   If we take $u = e^{2\pi i/5} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$, then $\frac{1}{u} = e^{-2\pi i/5} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}$, so $v = u + \frac{1}{u} = 2\cos\frac{2\pi}{5}$.
   The roots of $v^2 + v - 1 = 0$ are $\frac{-1 \pm \sqrt{5}}{2}$.
   Since $\cos\frac{2\pi}{5} > 0$ we must have $2\cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{2}$, so $\cos\frac{2\pi}{5} = \frac{-1 + \sqrt{5}}{4}$.