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1. (a) $2 - 3\mathrm{i} - 4 + 5\mathrm{i} = -2 + 2\mathrm{i}$

   (b) $6 - 9\mathrm{i} + 8\mathrm{i} - 12\mathrm{i}^2 = 6 - \mathrm{i} - 12(-1) = 18 - \mathrm{i}$

   (c) $(5 - \mathrm{i})(5 - \mathrm{i}) = 25 - 10\mathrm{i} + (-1) = 24 - 10\mathrm{i}$

   (d) $\dfrac{(6-2\mathrm{i})(3-4\mathrm{i})}{(3+4\mathrm{i})(3-4\mathrm{i})} = \dfrac{10-30\mathrm{i}}{25} = \dfrac{2}{5} - \dfrac{6}{5}\mathrm{i}$

2. (a) Modulus $= \sqrt{2}$, argument $= \arctan 1 = \dfrac{\pi}{4}$

   (b) Modulus $= 5$, argument $= \arctan(-4/3) \approx -0.927$ (in fourth quadrant)

   (c) Modulus $= \sqrt{29}$, argument $= \arctan(-2.5) \approx 1.95$ (in second quadrant)

   (d) Modulus $= 2$, argument $= \arctan \dfrac{1}{\sqrt{3}} = -\dfrac{5\pi}{6}$ (in third quadrant)

   (e) Modulus $= 7$, argument $= -\dfrac{\pi}{2}$ (obviously! It's on the negative imaginary axis)

3. (a) $x^2 = -\dfrac{1}{4}$ so $x = \pm\dfrac{1}{2}\mathrm{i}$

   (b) $x = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4\mathrm{i}}{2} = -1 \pm 2\mathrm{i}$

4. (a) Let $w = u + v\mathrm{i}$, $z = x + y\mathrm{i}$.
   Then $w + z = (u + x) + (v + y)\mathrm{i}$, so $\overline{w+z} = (u + x) - (v + y)\mathrm{i}$.
   Also $\bar{w} = u - v\mathrm{i}$, $\bar{z} = x - y\mathrm{i}$, so $\bar{w} + \bar{z} = u + x - v\mathrm{i} - y\mathrm{i} = (u + x) - (v + y)\mathrm{i}$.
   Hence $\overline{w+z} = \bar{w} + \bar{z}$.

   (b) Let $w = u + v\mathrm{i}$, $z = x + y\mathrm{i}$.
   Then $wz = (ux - vy) + (uy + vx)\mathrm{i}$, so $\overline{wz} = (ux - vy) - (uy + vx)\mathrm{i}$.
   Also $\bar{w} = u - v\mathrm{i}$, $\bar{z} = x - y\mathrm{i}$, so $\bar{w}\,\bar{z} = ux - uy\mathrm{i} - vx\mathrm{i} - vy = (ux - vy) - (uy + vx)\mathrm{i}$.
   Hence $\overline{wz} = \bar{w}\,\bar{z}$.

5. Method 1: Let $z = x + y\mathrm{i}$. Then $\dfrac{z}{\bar{z}} = \dfrac{(x+y\mathrm{i})(x+y\mathrm{i})}{(x-y\mathrm{i})(x+y\mathrm{i})} = \dfrac{x^2 - y^2 + 2xy\mathrm{i}}{x^2+y^2}$, whch has modulus r where $r^2 = \left(\dfrac{x^2-y^2}{x^2+y^2}\right)^2 + \left(\dfrac{2xy}{x^2+y^2}\right)^2$

   $$= \frac{(x^4 + y^4 - 2x^2y^2) + (4x^2y^2)}{(x^2+y^2)^2} = \frac{x^4+y^4+2x^2y^2}{x^4+y^4+2x^2y^2} = 1, \text{ so } r = 1.$$

   Method 2: We have shown that if $|z_1| = r_1$ and $|z_2| = r_2$ then $\dfrac{z_1}{z_2}$ has modulus $\dfrac{r_1}{r_2}$.
   Let $z = x + y\mathrm{i}$, so $|z| = \sqrt{x^2+y^2}$. Then $\bar{z} = x - y\mathrm{i}$, so $|\bar{z}| = \sqrt{x^2 + (-y)^2} = \sqrt{x^2+y^2}$.
   Hence $|z| = |\bar{z}|$, so $|z/\bar{z}| = 1$.

   We have also shown that $\arg(z_1/z_2) = \arg(z_1) - \arg(z_2)$.

   It is clear from an Argand diagram that $\arg(\bar{z}) = -\arg(z)$, so $\arg(z/\bar{z}) = \arg(z) - (-\arg(z)) = 2\arg(z)$.

   So $\arg(z/\bar{z})$ is the angle in the interval $-\pi$ to π corresponding to $2\arg(z)$, or equivalently $\arg(z^2)$.

6. Roots occur in conjugate pairs, so the other root is $2 - 3i$.

Equation is $(x - (2 + 3\mathrm{i}))(x - (2 - 3\mathrm{i})) = 0$.

Multiplying out gives $x^2 - 4x + 13 = 0$.

7. $a^2 - b^2 + 2ab\mathrm{i} = 5 + 12\mathrm{i}$, so $a^2 - b^2 = 5, ab = 6$.

$b = \dfrac{6}{a}$, so $a^2 - \dfrac{36}{a^2} = 5$. Multiply through by a^2 to get $a^4 - 5a^2 - 36 = 0$.

Factorise: $(a^2 + 4)(a^2 - 9) = 0$. Since a is real, $a^2 = 9$ so $a = \pm 3$.

Thus $a = 3, b = 2$ or $a = -3, b = -2$.

The two square roots of $5 + 12\mathrm{i}$ are $3 + 2\mathrm{i}$ and $-3 - 2\mathrm{i}$.

Now if $\tan 2\phi = 2.4$ then $2\phi = \arg(5 + 12\mathrm{i})$.

Thus $\phi = \arg(\sqrt{5 + 12\mathrm{i}}) = \arg(3 + 2\mathrm{i})$, so $\tan \phi = \dfrac{2}{3}$.

8. $z^4 = (\cos\theta)^4 + 4(\cos\theta)^3(\mathrm{i}\sin\theta) + 6(\cos\theta)^2(\mathrm{i}\sin\theta)^2 + 4(\cos\theta)(\mathrm{i}\sin\theta)^3 + (\mathrm{i}\sin\theta)^4$

$= \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta + \mathrm{i}(4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta)$.

By de Moivre's Theorem, $z^4 = \cos 4\theta + \mathrm{i}\sin 4\theta$.

Comparing real and imaginary parts,

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ and $\sin 4\theta = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$.

9. $z^n = \cos n\theta + \mathrm{i}\sin n\theta$ and $\dfrac{1}{z^n} = \cos n\theta - \mathrm{i}\sin n\theta$,

so $z - \dfrac{1}{z^n} = (\cos n\theta + \mathrm{i}\sin n\theta) - (\cos n\theta - \mathrm{i}\sin n\theta) = 2\mathrm{i}\sin n\theta$.

By the Binomial Theorem,

$(z - \dfrac{1}{z})^5 = z^5 - 5z^3 + 10z - \dfrac{10}{z} + \dfrac{5}{z^3} - \dfrac{1}{z^5} = (z^5 - \dfrac{1}{z^5}) - 5(z^3 - \dfrac{1}{z^3}) + 10(z - \dfrac{1}{z})$

$= 2\mathrm{i}\sin 5\theta - 5(2\mathrm{i}\sin 3\theta) + 10(2\mathrm{i}\sin\theta)$.

Also $(z - \dfrac{1}{z})^5 = (2\mathrm{i}\sin\theta)^5 = 32\mathrm{i}\sin^5\theta$.

Thus $32\mathrm{i}\sin^5\theta = 2\mathrm{i}\sin 5\theta - 10\mathrm{i}\sin 3\theta + 20\mathrm{i}\sin\theta$.

Dividing through by $2\mathrm{i}$ gives the stated result.

10. $z^3 - 1 = (z - 1)(z^2 + z + 1)$, so if $z^3 - 1 = 0$ then $z = 1$ or $z = \dfrac{-1 \pm \mathrm{i}\sqrt{3}}{2}$.

The complex roots $-\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}\mathrm{i}$ have modulus 1 and arguments $\pm\dfrac{2\pi}{3}$, so they are $\cos\dfrac{2\pi}{3} + \mathrm{i}\ \sin\dfrac{2\pi}{3}$ and $\cos\dfrac{2\pi}{3} - \mathrm{i}\ \sin\dfrac{2\pi}{3}$.