CM 1021 Mathematical Methods for Computing I

Exercise Sheet 4 - Solutions

1. The vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are defined as follows; $\mathbf{a} = (2, 3, 4), \mathbf{b} = (1, 5, 8), \mathbf{c} = (2, -1, -4)$. Calculate the following:

   - $||\mathbf{a}||, \quad ||\mathbf{c}||, \quad \mathbf{b} + \mathbf{c}, \quad \mathbf{a} - \mathbf{c}$
   - $\mathbf{a}.\mathbf{c}, \quad \mathbf{b}.(\mathbf{c} - \mathbf{a})$

2. One student bought 2 CD's and 3 DVD's for £65 while another student bought 4 CD's and 1 DVD for £55. How much does a CD and a DVD cost?

$$
\begin{aligned}
&2c + 3d = 65 \Longrightarrow 4c + 6d = 130\\
&4c + d = 55\\
&5d = 75 \Longrightarrow d = 15, \quad c = 10
\end{aligned}
$$

3. Solve the following systems of linear equations:

$$
(a) \quad \begin{aligned} x + 3y &= -1 \\ 4x - 2y &= 10 \end{aligned} \qquad\qquad (b) \quad \begin{aligned} 2x - 5y &= -8 \\ 4x + y &= 6 \end{aligned}
$$

$$
(c) \quad \begin{aligned} x + 2y + z &= -3 \\ x + 3y + 3z &= -9 \\ -x + 3y - 2z &= 17 \end{aligned} \qquad\qquad (d) \quad \begin{aligned} x + 2y + 5z &= 5 \\ 2x + y + 3z &= 8 \\ 3x + 2y + z &= 10 \end{aligned}
$$

(a)

$$
\begin{aligned}
&x + 3y = -1 \Longrightarrow 4x + 12y = -4\\
&4x - 2y = 10\\
&14y = -14 \Longrightarrow y = -1, \quad x = 2
\end{aligned}
$$

(b)

$$
\begin{aligned}
&2x - 5y = 8 \Longrightarrow 4x - 10y = -16\\
&4x + y = 6\\
&-11y = -22 \Longrightarrow y = 2, \quad x = 1
\end{aligned}
$$

(c) $$\left(\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ 1 & 3 & 3 & -9 \\ -1 & 3 & -2 & 17 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -6 \\ 0 & 5 & -1 & 14 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & -3 \\ 0 & 1 & 2 & -6 \\ 0 & 0 & -11 & 44 \end{array}\right)$$

Thus $z = -4$ and by substitution $y = 2$ and $x = -3$.

4. For the two matrices

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 4 \\ -2 & 5 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 4 & -2 & 3 \\ 1 & 0 & 3 \\ -1 & -2 & 3 \end{pmatrix},$$

find (i) $B - 3A$ (ii) AB and (iii) BA.

$$B - 3A = \begin{pmatrix} 1 & -8 & 0 \\ -8 & 3 & -9 \\ 5 & -17 & 0 \end{pmatrix} \qquad AB = \begin{pmatrix} 5 & -4 & 12 \\ 7 & -14 & 18 \\ -4 & 2 & 12 \end{pmatrix}$$

$$BA = \begin{pmatrix} -8 & 25 & 1 \\ -5 & 17 & 4 \\ -13 & 15 & 6 \end{pmatrix}$$

5. For the matrix

$$A = \begin{pmatrix} 1 & 3 & 1 \\ -2 & 5 & 3 \\ -3 & 0 & 1 \end{pmatrix},$$

show that the matrix

$$A^{-1} = \begin{pmatrix} -5 & 3 & -4 \\ 7 & -4 & 5 \\ -15 & 9 & -11 \end{pmatrix}$$

is the inverse of A and hence find the solution of the following system of linear equations:

$$\begin{pmatrix} 1 & 3 & 1 \\ -2 & 5 & 3 \\ -3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 11 \\ 1 \end{pmatrix}.$$

$$A \times A^{-1} = \begin{pmatrix} -5 + 21 - 15 & 3 - 12 + 9 & -4 + 15 - 11 \\ 10 + 35 - 45 & -6 - 20 + 27 & 8 + 25 - 33 \\ 15 + 0 - 15 & 9 + 0 - 9 & 12 + 0 - 11 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We can solve the system directly to give

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \times \begin{pmatrix} 6 \\ 11 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ -2 \end{pmatrix}$$

6. Show that the following linear system of equations does not have a solution: (Hint: Use Gaussian elimination)

$$\begin{aligned} x + 2y + z &= 4 \\ 2x + 5y - 2z &= 6 \\ x + 4y - 7z &= 6 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 2 & 5 & -2 & 6 \\ 1 & 4 & -7 & 6 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -4 & -2 \\ 0 & 2 & -8 & 2 \end{array}\right) \Longrightarrow \left(\begin{array}{ccc|c} 1 & 2 & 1 & 4 \\ 0 & 1 & -4 & -2 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

The final row implies that $0 = 6$ which is nonsense. We can see that $\rho(A) \neq \rho(A|b$ using the terminology in the lectures. The system is inconsistent and thus has no solution. *(Geometrically this means that the three planes represented by the three equations do not have a common point of intersection)*

7. Solve the following system of equations

$$\begin{aligned} 2u + v - 2w + 3x &= 2 \\ v + 3w - 3x &= 1 \\ 2u + 3v + 2w - x &= 0 \\ -4u - 3v + 5w - 4x &= 0 \end{aligned}$$

First we use Gaussian elimination, note that we must always ensure that the the first non-zero entry in each row is 1 so we divide through the whole row by the first non-zero entry as necessary to achieve this

$$\left(\begin{array}{cccc|c} 2 & 1 & -2 & 3 & 2 \\ 0 & 1 & 3 & -3 & 1 \\ 2 & 3 & 2 & -1 & 0 \\ -4 & -3 & 5 & -4 & 0 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -1 & \frac{3}{2} & 1 \\ 0 & 1 & 3 & -3 & 1 \\ 0 & 2 & 4 & -4 & 2 \\ 0 & -1 & 1 & 2 & 4 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -1 & \frac{3}{2} & 1 \\ 0 & 1 & 3 & -3 & 1 \\ 0 & 0 & -2 & 2 & -4 \\ 0 & 0 & 4 & -1 & 5 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -1 & \frac{3}{2} & 1 \\ 0 & 1 & 3 & -3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 4 & -1 & 5 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -1 & \frac{3}{2} & 1 \\ 0 & 1 & 3 & -3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 3 & -3 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -1 & \frac{3}{2} & 1 \\ 0 & 1 & 3 & -3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

We have $x = -1$ and we could find the other variables by substitution. Alternatively we can continue with the row operations as follows (this is called Gauss-Jordan elimination)

$$\left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -1 & \frac{3}{2} & 1 \\ 0 & 1 & 3 & -3 & 1 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & -1 & 0 & \frac{5}{2} \\ 0 & 1 & 3 & 0 & -2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right) \Longrightarrow \left(\begin{array}{cccc|c} 1 & \frac{1}{2} & 0 & 0 & \frac{7}{2} \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

$$\Longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right)$$

Thus $u = 6, v = -5, w = 1, x = -1$.

8. Find the inverse of the matrix

$$\begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix},$$

and hence solve the system

$$3x + 4y = 13, \quad -2x + 5y = -1.$$

$$\begin{pmatrix} 3 & 4 \\ -2 & 5 \end{pmatrix}^{-1} = \frac{1}{15+8}\begin{pmatrix} 5 & -4 \\ 2 & 3 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{23}\begin{pmatrix} 5 & -4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 13 \\ -1 \end{pmatrix} = \frac{1}{23}\begin{pmatrix} 69 \\ 23 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

9. Find the eigenvalues and eigenvectors of the following matrices

(i) $\begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$,(ii) $\begin{bmatrix} 3 & 4 \\ 1 & 6 \end{bmatrix}$, (iii) $\begin{bmatrix} 4 & 1 \\ 6 & 3 \end{bmatrix}$

(i) $-1, (2,1)$ and $6, (3,2)$
(ii) $7, (1,1)$ and $2, (-4,1)$
(iii) $6, (1,2)$ and $1, (1,-3)$

More Challenging Questions

10. Find two non-zero 2×2 matrices A and B such that their product is the zero matrix

There are many examples e.g.

$$\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & -6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

11. Is $(A+B)(A-B) = A^2 - B^2$ for matrices A and B?

$(A+B)(A-B) = AA - AB + BA - BB = A^2 - B^2 - AB + BA$ but $AB \neq BA$
so $(A+B)(A-B) \neq A^2 - B^2$ unless A and B commute

12. Let

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

Find A^2, A^3 and A^4. What is A^n for positive integer n?

$$A^2 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad A^3 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \quad A^4 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$$

The sequence suggests that $A^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$

13. Find the eigenvalues and eigenvectors of the following matrices

(i) $\begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$, (ii) $\begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$

(i) The characteristic equation is $\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$ giving eigenvalues and eigenvectors $1, (0,1,0)$, $3, (-1,1,1)$, $2, (1,-2,-2)$
(ii) The characteristic equation is $\lambda^3 + 5\lambda^2 - 42\lambda - 144 = 0$, which factors to give $(\lambda+8)(\lambda-6)(\lambda+3)$ giving eigenvalues and eigenvectors $8, (1,2,-3)$, $6, (-5,4,1)$, $-3, (1,1,1)$