# CM 1021 Mathematical Methods for Computing I

## Exercise Sheet 7 Solutions

1. Find the value of

$$\text{(a)} \quad \sum_{\mathrm{r=1}}^{23} \mathrm{r} \qquad \text{(b)} \quad \sum_{\mathrm{r=7}}^{13} (3\mathrm{r} - 2)$$

(a)

$$\sum_{r=1}^{23} r = \frac{1}{2} \times 23 \times (1 + 23) = 276$$

(b)

$$\sum_{r=7}^{13} (3r - 2) = 3 \sum_{r=7}^{13} (3r) - 14 = 3 \times \frac{7}{2}(13 + 7) = 196.$$

2. Find the sum of all even positive integers less than 2000.

We want twice the sum of all numbers from 1 to 999, (i.e. 2 to 1998)

$$2 \sum_{r=1}^{999} r = 2 \times \frac{1}{2} \times (1 + 999) \times 999 = 999,000$$

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3. The boring of a well costs £5 for the first metre depth, £11 for the second and £17 for the third. The costs for each successive metre continue in the same arithmetic series. Find the cost of boring a well of (a) 50 metres and (b) 100 metres

$a = 5, \quad a + d = 11, \quad d = 6.$
Then $u_{50} = 5 + 49 \times 6 = 299$ and $u_{100} = 5 + 99 \times 6 = 599$.

Hence $S_{50} = 50 \times \frac{1}{2} \times (299 + 5) = 7600$ while $S_{100} = 100 \times \frac{1}{2} \times (599 + 5)$.

4. Find the 10th term and the sum of the first ten terms of the following geometric series:

$$a + ar + ar^2 + ar^3 \ldots$$

when

- (i) $a = 10, r = 2$ (ii) $a = 10, r = -2$ (iii) $a = 10, r = -\frac{1}{2}$
- (iv) $a = 3, r = 6$ (v) $a = 3, r = -6$.

$S_i = 10 \dfrac{1 - 2^{10}}{1 - 2} = 10,230, \quad u_{10} = 5120.$

$$S_{ii} = 10\frac{1-(-2)^{10}}{1-(-2)} = -3410, \quad u_{10} = -5120.$$

$$S_{iii} = 10\frac{1-\left(\frac{1}{2}\right)^{10}}{1-\left(\frac{1}{2}\right)} = \frac{5115}{256} \quad u_{10} = \frac{5}{256}.$$

$$S_{iv} = 3\frac{1-6^{10}}{1-6} = 36,279,705, \quad u_{10} = 30,233,088.$$

$$S_v = 3\frac{1-(-6)^{10}}{1-(-6)} = -25,914,075, \quad u_{10} = -30,233,088.$$

5. Find the sum to $n$ terms and the sum to infinity (where the series converges) of the following series:

- $\frac{3}{10} + \frac{3}{100} + \frac{3}{1000} + \ldots$

  Here $a = \frac{3}{10}$ and $r = \frac{1}{10}$. Thus

$$S_n = \frac{3}{10}\frac{1-\left(\frac{1}{10}\right)^n}{1-\left(\frac{1}{10}\right)} \Longrightarrow S_\infty = \frac{\frac{3}{10}}{1-\left(\frac{1}{10}\right)} = \frac{1}{3}.$$

- $16 - 8 + 4 - \ldots$

  We have $a = 16$ and r=$-\frac{1}{2}$

$$S_n = 16\frac{(1-\left(-\frac{1}{2}\right)^n}{1-\left(-\frac{1}{2}\right)} \Longrightarrow S_\infty = \frac{16}{\frac{3}{2}} = \frac{32}{3}.$$

- $8 - 12 + 18 - \ldots$

  $a = 8, r = -\frac{3}{2}$ so the series will not converge, in fact it will oscillate to infinity.

$$S_n = 8\frac{(1-\left(-\frac{3}{2}\right)^n}{1-\left(-\frac{1}{2}\right)} = \frac{16}{5}\left(1-\left(-\frac{3}{2}\right)^n\right)$$

6. Find the first 3 terms in the expansion of the following in **ascending** powers of $x$

$$(a) \quad (1-x)^{23} \qquad (b) \quad (1+x)^{15}$$

7. Find the first three terms in **descending** powers of $x$ of $(5x-3)^7$.

8. Expand $(1+x)^{10}$, in ascending powers of $x$ obtaining the first four terms. Hence obtain an approximation of $0.998^{10}$

9. Expand the following in series in ascending powers of $x$ up to and including the term in $x^3$, simplifying as much as possible and stating the range of values of $x$ for which the series is valid:

$$(a)\quad \frac{1}{(1+x)^2} \quad (b)\quad \frac{1}{(1-x)^3} \quad (c)\quad \frac{2}{\left(1-\frac{x}{2}\right)^2} \quad (d)\quad (4+3x)^{\frac{1}{2}}$$

$$(e)\quad (1-x)^{\frac{3}{4}} \quad (f)\quad \frac{1}{(100+x)^{\frac{1}{2}}} \quad (g)\quad (1-3x)^{\frac{1}{3}}\,.$$

10. Find the first 3 terms in the expansion of the following in **ascending** powers of $x$

$$(a)\quad (1-x)^{23} \qquad (b)\quad (1+x)^{15}$$

(a)

$$(1-x)^{23} = 1 + 23(-x) + \frac{23\times 22}{2}(-x)^2 \ldots \quad = 1 - 23x + 253x^2 \ldots$$

(b)

$$(1+x)^{15} = 1 + 15x + 105x^2 \ldots$$

11. Find the first three terms in **descending** powers of $x$ of $(5x-3)^7$.

$$(5x-3)^7 = (5x)^7 + 7(5x)^6\times(-3) + \frac{7\times 6}{2}(5x)^5\times(-3)^2 \ldots = 78125x^7 - 328,125x^6 + 590,625x^5 \ldots$$

12. Expand $(1+x)^{10}$, in ascending powers of $x$ obtaining the first four terms. Hence obtain an approximation of $0.998^{10}$

$$(1+x)^{10} = 1 + 10x + 45x^2 + 120x^3 \ldots$$

Let $x = 0.002$, then $0.998^{10} = 1.0 - 0.020 + 0.0180 - 0.0000096 = 0.9801790400$.
Maple gives $0.998^{10} = 0.9801790434$

13. Expand the following in series in ascending powers of $x$ up to and including the term in $x^3$, simplifying as much as possible and stating the range of values of $x$ for which the series is valid:

$$(a)\quad \frac{1}{(1+x)^2} \quad (b)\quad \frac{1}{(1-x)^3} \quad (c)\quad \frac{2}{\left(1-\frac{x}{2}\right)^2} \quad (d)\quad (4+3x)^{\frac{1}{2}}$$

$$(e)\quad (1-x)^{\frac{3}{4}} \quad (f)\quad \frac{1}{(100+x)^{\frac{1}{2}}} \quad (g)\quad (1-3x)^{\frac{1}{3}}\,.$$

(a)

$$\frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 \ldots \qquad |x| < 1$$

(b)

$$\frac{1}{(1-x)^3} = 1 + 3x + 6x^2 + 10x^2 \ldots \qquad |x| < 1$$

(c)

$$\frac{2}{\left(1-\frac{x}{2}\right)^2} = 2 + 2x + \frac{3}{2}x^2 + x^3 \ldots \qquad |x| < 2$$

(d)

$$\sqrt{4+3x} = 2 + \frac{3x}{4} - \frac{9x^2}{64} + \frac{27x^3}{512} \ldots |x| < \frac{4}{3}$$

(e)

$$(1-x)^{\frac{3}{4}} = 1 - \frac{3x}{4} - \frac{3x^2}{32} - \frac{5x^3}{128} \ldots \qquad |x| < 1$$

(g)

$$\frac{1}{\sqrt{100+x}} = \frac{1}{10} - \frac{1}{2000}x + \frac{3}{800000}x^2 - \frac{1}{32000000}x^3 \ldots \qquad |x| < 100$$

(h)

$$\sqrt[3]{1-3x} = 1 - x - x^2 - \frac{5x^3}{3} \ldots \qquad |x| < \frac{1}{3}$$

**More Challenging Questions**

14. Find the sum of all the integers between 0 and 200 that are not divisible by 4.

We want the sum of the integers from 1 to 200 inclusive minus four times the sum of the numbers from 1 to 50 inclusive.(i.e. 4,8,12...200)

$$\sum_{r=1}^{50} r = \frac{50 \times (50+1)}{2} = 1275, \qquad \sum_{r=1}^{200} r = \frac{200 \times (200+1)}{2} = 20,100.$$

The solution is thus $20,100 - 4 \times 1275 = 15,000$.

15. The first and third term of an arithmetic series are $\frac{2}{3}$ and $\frac{3}{2}$ respectively. Find the common difference and the sum of the first twelve terms.

$a = \frac{2}{3}, \quad a + 2d = \frac{3}{2}$. Solving simultaneously we have $a = \frac{5}{6}$ and $d = \frac{5}{12}$.
Then the twelfth term is $u_{12} = \frac{2}{3} + 11 \times \frac{5}{12} = \frac{63}{12}$ and the sum to twelve terms is

$$S_{12} = \frac{12}{2}\left(\frac{2}{3} + \frac{63}{12}\right) = \frac{71}{2}.$$

16. Find the value of

$$\text{(a)} \quad \sum_{\mathrm{n}=4}^{9} (1.5)^{\mathrm{n}} \qquad \text{(b)} \quad \sum_{\mathrm{n}=2}^{10} 4\left(\frac{3}{4}\right)^{\mathrm{n}-1}$$

$$S_a = \sum_{n=0}^{9} (1.5)^n - \sum_{n=0}^{4} (1.5)^n = \frac{3}{2}\frac{1-\left(\frac{3}{2}\right)^{10}}{1-\left(\frac{3}{2}\right)} - \frac{3}{2}\frac{1-\left(\frac{3}{2}\right)^{4}}{1-\left(\frac{3}{2}\right)}$$

$$= 3\left(\left(\frac{3}{2}\right)^{10} - \left(\frac{3}{2}\right)^{4}\right) = \frac{53865}{512}$$

$$S_b = \sum_{m=1}^{9} 4 * \left(\frac{3}{4}\right)^m \qquad \text{substituting } m = n-1$$

$$= 4\frac{1-\left(\frac{3}{4}\right)^{10}}{1-\frac{3}{4}} - 4 = \frac{727383}{65536}$$

17. Suppose that the price of a house increases at a constant 3% per annum. At the start of 1994 it was worth £45,000. What was the value at the start of 1980? What would the value be at the start of 2020? 2200?

$$u_0 = \frac{45,000}{(1.03)^{14}} = £29,750$$

$$u_{26} = 45,000 \times 1.03^26 = £97,047$$

$$u_{206} = 45,000 \times 1.03^206 = £19,846,357$$

18. The coefficients of $x, x^2$ and $x^3$ in the expansion of $(1+x)^n$ form the first three terms of an arithmetic series. Show that $n = 7$.

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3$$

If the coefficients of $x, x^2$ and $x^3$ are in AP then

$$\frac{n(n-1)}{2} - n = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2}$$

$$n(n-1) - n = \frac{n^3 - 3n^2 + 2n}{6} \qquad\qquad 1 - 2px + x^2\left(-2q + 3p^2\right)\ldots$$

$$6n^2 - 12n = n^3 - 3n^2 + 2n$$

either $n = 0$ or

$$n^2 - 9n + 14 = 0$$

$$(n-2)(n-7) = 0$$

Thus we must have $n = 7$ since $n = 2$ gives us no cubic term in the expansion.

19. Find the coefficient of $x^{12}$ in the expansion of $(x+y)^{18}$. Evaluate this term with $x = 2, y = \frac{1}{3}$.

The required term is $\left(\begin{array}{c} 18 \\ 12 \end{array}\right) x^{12}y^6$

Making the substitution $x = 2, y = \frac{1}{3}$ we have $\dfrac{25346048}{243} = 10430.47243$

20. Given that $|x| < 1$, expand $\left(\dfrac{1+x}{1-x}\right)^{\frac{1}{3}}$ up to and including the term in $x^2$

$$\begin{aligned}
\left(\frac{1+x}{1-x}\right)^{\frac{1}{3}} &= \left(1 + \frac{x}{3} - \frac{x^2}{9}\dots\right)\left(1 + \frac{x}{3} + \frac{2x^2}{9}\dots\right) \\
&= 1 + \frac{x}{3} + \frac{2x^2}{9}\dots \\
&\qquad + \frac{x}{3} + \frac{x^2}{9}\dots \\
&\qquad\qquad - \frac{x^2}{9}\dots \\
&= 1 + \frac{2x}{3} + \frac{2x^2}{9}\dots
\end{aligned}$$

21. The coefficients of $x$ and $x^2$ in the expansion of $(1+px+qx^2)^{-2}$ in ascending powers of $x$ are 4 and 14. Find $p$ and $q$

$$\begin{aligned}
(1+pq+qx^2)^{-2} &= 1 - 2(px+qx^2) + 3(px+qx^2)^2 \\
&= 1 - 2px - 2qx^2 + 3p^2x^2 + O(x^3) \\
&= 1 - 2px + x^2\left(-2q + 3p^2\right)\dots
\end{aligned}$$

Hence $-2p = 4, \quad p = 2$ and $-2q + 3p^2 = 14, \quad q = -1$