CM 1021 Mathematical Methods for Computing I

Exercise Sheet 4a Solutions

1. Let $\mathbf{u} = (-3, 2, 1, 0)$, $\quad \mathbf{v} = (4, 7, -3, 2)$, $\quad \mathbf{w} = (5, -2, 8, 1)$, $\quad \mathbf{x} = (1, 2, 3, 4)$

   (a) Find (i) $\mathbf{v} - \mathbf{w}$ (ii) $3\mathbf{w} + 2\mathbf{x} - \mathbf{u}$ (iii) $(6\mathbf{w} + 2\mathbf{v}) - (4\mathbf{x} - \mathbf{u})$

   (i) $v - w = (-1, 9, -11, 1)$ (ii) $3\mathbf{w} + 2\mathbf{x} - \mathbf{u} = (20, -4, 29, 11)$
   (iii) $(6\mathbf{w} + 2\mathbf{v}) - (4\mathbf{x} - \mathbf{u}) = (37, -4, 29, 11)$

   (b) Find (i) $||\mathbf{u}||$ (ii) $||\mathbf{w}+\mathbf{x}||$ (iii) $||\mathbf{x}-\mathbf{u}||$ (iv) $\hat{\mathbf{w}}$ (v) $\widehat{2\mathbf{v} - \mathbf{x}}$

   (i) $||\mathbf{u}|| = \sqrt{14}$ (ii) $||\mathbf{w}+\mathbf{x}|| = \sqrt{182}$ (iii) $||\mathbf{x}-\mathbf{u}|| = \sqrt{34}$
   (iv) $\hat{\mathbf{w}} = \frac{1}{\sqrt{94}}(5, -2, 8, 1)$ (v) $\widehat{2\mathbf{v} - \mathbf{x}} = \frac{1}{\sqrt{274}}(7, 12, -9, 0)$

   (c) Find (i) $\mathbf{u} \cdot \mathbf{v}$ (ii) $\mathbf{w} \cdot \mathbf{x}$ (iii) $d(\mathbf{w}, \mathbf{v})$ (iv) $d(\mathbf{u}, \mathbf{x})$

   (i) $\mathbf{u} \cdot \mathbf{v} = -1$ (ii) $\mathbf{w} \cdot \mathbf{x} = 29$ (iii) $d(\mathbf{w}, \mathbf{v}) = \sqrt{205}$ (iv) $d(\mathbf{u}, \mathbf{x}) = 6$

   (d) Find a vector $\mathbf{y}$ such that $5\mathbf{y} - 2\mathbf{v} = 2(\mathbf{w} - 5\mathbf{x})$

   $\mathbf{y} = (1.6, -2, -4, -6.4)$

   (e) Find the value of all scalars k such that $||k\mathbf{u}|| = 14$

   $\sqrt{14k} = 14 \rightarrow k = \pm\sqrt{14}.$

   (f) Are $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ linearly independent?

   The most straightforward method is to perform row operations on the matrix consisting of each of these vectors as a row vector

   $$\begin{pmatrix} -3 & 2 & 1 & 0 \\ 4 & 7 & -3 & 2 \\ 5 & 2 & 8 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & \frac{5}{4} & \frac{3}{2} \\ 0 & 0 & 1 & \frac{10}{11} \\ 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$ We see that the rank of this matrix is 4 so the vectors are linearly independent

2. Which of the following pairs of vectors are orthogonal?

   (a) $\mathbf{u} = (-1, 3, 2) \quad \mathbf{v} = (4, 2, -1)$

(b) $\mathbf{u} = (-2, -2, -2)$ $\mathbf{v} = (1, 1, 1)$

(c) $\mathbf{u} = (-4, 6, 10, 2)$ $\mathbf{v} = (2, 1, -2, 11)$

(a) $\mathbf{u} \cdot \ \mathbf{v} = 0$ orthogonal (b) $\mathbf{u} \cdot \mathbf{v} = -9$ not orthogonal (c) $\mathbf{u} \cdot \mathbf{v} = 0$ orthogonal.

3. For what value of k are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

   (a) $\mathbf{u} = (2, 1, 3)$ $\mathbf{v} = (1, 7, k)$

   (b) $\mathbf{u} = (k, k, 1)$ $\mathbf{v} = (k, 5, 6)$

   (a) $2 + 7 + 3k = 0 \to k = -3$ (b) $k^2 + 5k + 6 = 0 \to k = -2$ and $k = -3$

4. Which of the following are linear combinations of $\mathbf{u} = (1, -2, 3)$ and $\mathbf{v} = (2, -1, 4)$

   (a) $(-4, -1, 6)$

   (b) $(0, -3, 2)$

   (c) $(4, -3, 8)$

   (d) $(3 - 5, 9)$

   Either by inspection or by using row operations on the matrices

$$(a) \quad \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ -4 & -1 & 6 \end{pmatrix} \quad (b) \quad \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ 0 & -3 & 2 \end{pmatrix} \quad (c) \quad \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ 4 & -3 & 8 \end{pmatrix} \quad (d) \quad \begin{pmatrix} 1 & -2 & 3 \\ 2 & -1 & 4 \\ 3 & 5 & -9 \end{pmatrix}$$

   We find that in (a) $2\mathbf{u} - 3\mathbf{v} = (-4, -1, 6)$, in (b) $2\mathbf{u} - \mathbf{v} = (0, -3, 2)$ but (c) and (d) are not linear combination of $\mathbf{u}$ and $\mathbf{v}$.

5. Express the following as linear combinations of
   $\mathbf{u} = (2, 1, 4), \mathbf{v} = (1, -1, 3)$ and $\mathbf{w} = (3, 2, 5)$

   (a) $(-9, -7, -15)$

   (b) $(6, 11, 6)$

   (c) $(1, 1, 1)$

   (d) $(7, 8, 9)$

   In each case we need to solve a system of three equations (either by Gaussian elimination or, much more quickly, using Maple for x_1, x_2, x_3.
   (a)

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= -9 \\ x_1 - x_2 + 2x_3 &= -7 \\ 4x_1 + 3x_2 + 5x_3 &= -15 \end{aligned}$$

$$(-9, -7, -15) = -2\mathbf{u} + \mathbf{v} - 2\mathbf{w}$$

(b)

$$\begin{aligned}2x_1 + x_2 + 3x_3 &= 6\\ x_1 - x_2 + 2x_3 &= 11\\ 4x_1 + 3x_2 + 5x_3 &= 6\end{aligned}$$

$$(6,11,6) = 4\mathbf{u} - 5\mathbf{v} + \mathbf{w}$$

(c)

$$\begin{aligned}2x_1 + x_2 + 3x_3 &= 1\\ x_1 - x_2 + 2x_3 &= 1\\ 4x_1 + 3x_2 + 5x_3 &= 1\end{aligned}$$

$$(1,1,1) = -\mathbf{u} + \mathbf{w}$$

(d)

$$\begin{aligned}2x_1 + x_2 + 3x_3 &= 7\\ x_1 - x_2 + 2x_3 &= 8\\ 4x_1 + 3x_2 + 5x_3 &= 9\\ (7,8,9) &= 19\mathbf{u} - 4\mathbf{v} - 11\mathbf{w}\end{aligned}$$

6. Determine whether or not these sets of vectors span $\mathbb{R}^3$:

   (a) $(2,2,2),(0,0,3),(0,1,1)$

   (b) $(2,-1,3),(4,1,2),(8,-1,8)$

   (c) $(3,1,4),(2,-3,5),(5,-2,9)$

   There are three vectors each with three components so a priori they are candidates for a spanning set. We need to establish whether or not they are linearly independent. We can do this by computing the rank of the matrix formed with each vector as a row vector, using row operations

$$(a)\begin{pmatrix}-2 & 2 & 2\\ 0 & 0 & 3\\ 0 & 1 & 1\end{pmatrix}\quad (b)\begin{pmatrix}2 & -1 & 3\\ 4 & 1 & 2\\ 8 & -1 & 8\end{pmatrix}\quad (c)\begin{pmatrix}3 & 1 & 4\\ 2 & -3 & 53\\ 5 & -2 & 9\end{pmatrix}$$

   (a) rank=3, spanning set, (b) rank=2, not a spanning set, (c) rank=3, spanning set,

7. Show that the following sets of vectors are linearly independent.

   (a) $(3,8,7,-3),(1,5,3,-1),(2,-1,2,6),(1,4,0,3)$

   (b) $(0,0,2,2),(3,3,0,0),(1,1,0,1),(2,3,1,4)$

   (c) $(0,3,-3,-6),(-2,0,0,6),(0,-4,-2,-2),(0,-8,4,-4)$

   (d) $(3,0,-3,-6),(0,2,3,1),(0,-2,-2,0),(-2,1,2,1)$

We need to establish the rank of the matrices formed by the vectors written as row vectors.

$$(a)\begin{pmatrix} 3 & 8 & 7 & -3 \\ 1 & 5 & 3 & -1 \\ 2 & -1 & 2 & 6 \\ 1 & 4 & 0 & 3 \end{pmatrix} \quad (b)\begin{pmatrix} 0 & 0 & 2 & 2 \\ 3 & 3 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 3 & 1 & 4 \end{pmatrix}$$

$$(c)\begin{pmatrix} 0 & 3 & -3 & 6 \\ 2 & 0 & 0 & 6 \\ 0 & -4 & -2 & -2 \\ 0 & -8 & 4 & -4 \end{pmatrix} \quad (d)\begin{pmatrix} 3 & 0 & -3 & -6 \\ 0 & 2 & 3 & 1 \\ 0 & -2 & -2 & 0 \\ -2 & 1 & 2 & 1 \end{pmatrix}$$

In each case the matrix has rank =4.

More challenging questions

8. Prove that for any vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ the vectors $\mathbf{u} - \mathbf{v}$, $\quad \mathbf{v} - \mathbf{w}$ and $\mathbf{w} - \mathbf{u}$ form a linearly dependent set.
   If we add the vectors $(\mathbf{u}-\mathbf{v})+(\mathbf{v}-\mathbf{w})+(\mathbf{w}-\mathbf{u}) = \mathbf{0}$. Thus non-zero scalars $(1, 1, 1)$ applied to the three vectors give the zero vector which is the definition of linear dependence.

9. Find the values of λ such that the following vectors form an independent set in $\mathbb{R}^3$
   $$v_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \quad v_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \quad v_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$
   If we carry out row operations on the matrix formed from the three vectors as its rows we obtain
   $$\begin{pmatrix} -\frac{1}{2} & \lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \\ \lambda & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{2} & \lambda & -\frac{1}{2} \\ 0 & -\frac{1}{2} - \lambda & \lambda + \frac{1}{2} \\ 0 & -\frac{1}{2} + 2\lambda^2 & -\frac{1}{2} - \lambda \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{2} & \lambda & -\frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 2\lambda^2 - \lambda - 1 \end{pmatrix}$$
   provided the third entry in the third row is non-zero the matrix has rank=3 and the vectors are linearly independent.
   Solving $2\lambda^2 - \lambda - 1 = 0$ we have $\lambda = 1$ and $\lambda = -\frac{1}{2}$. Providing that $\lambda \in \{\mathbb{R} \setminus \{-\frac{1}{2}, 1\}\}$ the set is linearly independent.

10. What geometric properties must a set of three vectors have if they are to span $\mathbb{R}^3$?
    The three vectors must be linearly independent, this means that no more than two of them can lie in the same plane

11. Determine whether the two lines
    $$r = (3, 2, 3, -1) + t(4, 6, 4, -2), \quad r = (0, 3, 4, 5) + t(1, -3, -4, -2)$$
    intersect in $\mathbb{R}^4$
    This is trivial, we need a consistent set of equations for t in each of the four components.

The fourth component gives us $-1-2t = 5 = 2t$ which implies $-1 = 5$ which is nonsense. The lines thus do not intersect.