CM 1021 Mathematical Methods for Computing I

Exercise Sheet 3 - solutions

1. Solve the following pairs of simultaneous equations

$$\begin{array}{ll} \text{(a)} & 3x + y = 7, 2x - 3y = 1 \\ \text{(b)} & 5x + 10y = 18, 2x - 4y = 6 \\ \text{(c)} & 2x + 3y = 16, x - y = 10 \end{array}$$

   (a) $x = 2, y = 1$ (b) $x = \frac{33}{10}, y = \frac{3}{20}$ (c) $x = \frac{46}{5}, y = -\frac{4}{5}$

2. Make x the subject of the following equations

$$\begin{array}{lll} \text{(a)} & y = \dfrac{3x+4}{4x+3} & x = \dfrac{4-3y}{4y-3} \\ \text{(b)} & y = \dfrac{x^2-4}{x^2+4} & x = 2\sqrt{\dfrac{1+y}{1-y}} \end{array}$$

3. Solve the following quadratic equations

$$\begin{array}{lll} \text{(a)} & x^2 - 11x + 10 = 0 & \{1, 10\} \\ \text{(b)} & x^2 + x - 3 = 0 & \{-\frac{1}{2} \pm \frac{1}{2}\sqrt{13}\} \\ \text{(c)} & 5x - 6x^2 - 1 = 0 & \{\frac{1}{3}, \frac{1}{2}\} \\ \text{(d)} & x^2 - 3x - 1 = 0 & \{-\frac{3}{2} \pm \frac{1}{2}\sqrt{13}\} \\ \text{(e)} & x^4 + 3x^2 - 4 = 0 & \{-1, 1\} \end{array}$$

4. (a) $x - 3 \leq 4 - \dfrac{x}{2}$ $\qquad \dfrac{3x}{2} \leq 7 \implies x \leq \dfrac{14}{3}$

   (b) $3x + 1 < -2 - x$ $\qquad x < -\dfrac{3}{4}$

   (c) $x^2 - 16 < 0$ $\qquad x^2 < 16 \implies x < -4$ and $x < 4$ or $x \in (-4, 4)$.

   (d) $x^2 + 4x + 3 > 0$ [Hint: factorise the left hand side and sketch a graph]
   $(x + 1)(x + 3) > 0 \implies x < -3$ and $x > -1$.

5. Sketch the region of the x, y plane corresponding to (a) $y > x + 2$ and $x + y \leq 5$.

6. For what range of values of c does the equation $3x^2 - 4x - c = 0$ have at least one real root?

   The discriminant must be non-negative. $(-4)^2 - 4 \times 3c \geq 0 \implies 16 - 12c \geq 0 \implies c \leq \dfrac{4}{3}$

More Challenging Questions:

7. Make x the subject of the following formulae and determine the range of values of y for which they are defined.

(a) $y = \dfrac{2x^2+1}{4x^2+3}$ $\qquad x = \dfrac{1}{2}\sqrt{\dfrac{2(1-3y}{2y-1}}$ $\quad y \in \left[\frac{1}{3}, \frac{1}{2}\right)$

(b) $y = \dfrac{2x^2+1}{4x+3}$ $\qquad x = y + \dfrac{1}{2}\sqrt{4\,y^2+6\,y-2}$ $\quad y \in \mathbb{R}$

(c) $y = \dfrac{e^{x^2}+1}{e^{x^2}-1}$ $\qquad x = \sqrt{\ln\left(\dfrac{y+1}{y-1}\right)}$

we must have the argument of the logarithm positive, i.e. $y < -1$ or $y > 1$

and the logarithm itself non-negative, thus $y > 1$ only

8. Solve the following equations

(a) $e^x + 21e^{-x} - 10 = 0$ $\qquad \{\ln 3, \ln 7\}$
(b) $3^{2x} - 5.3^{x+1} + 36 = 0$ $\qquad \{1, 1 + \frac{\ln 4}{\ln 3}\}$

9. The hyperbolic cosine (cosh) and the hyperbolic sine (sinh – pronounced "shine") are defined by

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \sinh x = \frac{1}{2}(e^x - e^{-x}).$$

(a) Are $\cosh x$ and $\sinh x$ even, odd or neither?

Cosh is an even function and sinh is odd

(b) Simplify the following expressions as far as possible.

(a) $2\cosh(\ln x)$, (b) $\cosh 5x + \sinh 5x$, (c) $\sinh(2\ln x)$.

(a) $\dfrac{1}{2}\left(x + \dfrac{1}{x}\right)$
(b) e^{5x}
(c) $\dfrac{1}{2}\left(x^2 - \dfrac{1}{x^2}\right)$

(c) Solve $\cosh x = 2$

$\dfrac{1}{2}(e^x + e^{-x}) = 2 \Longrightarrow e^x + e^{-x} = 4 \Longrightarrow e^{2x} - 4x^x + 1 = 0$
Now let $e^x = y$, then $y^2 - 4y + 1 = 0 \Longrightarrow y = 2 \pm \sqrt{3}$
So $e^x = 2 \pm \sqrt{3}$, so $x = \ln(2 + \sqrt{3})$ and $x = \ln(2 - \sqrt{3})$

10. Solve $x^3 - 6x^2 + 11x - 6 = 0$ [Hint: look for a factor]

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$f(1) = 0$ so we know that $x - 1$ is a factor. We can try other values to spot another root or we can factorise the expression.

$x^3 - 6x^2 + 11x - 6 = x^3 - x^2 - 5x^2 + 5x + 6x - 6 = x^2(x-1) - 5x(x-1) + 6(x-1) = (x-1)(x^2 - 5x + 6) = (x-1)(x-2)(x-3)$

thus $x = 1, 2, 3$