CM 1021 Mathematical Methods for Computing I

Exercise Sheet 1 - solutions

1. If $f(x) = x^2 + 2x$ and $g(x) = x - 1$, find $f(g(x))$ and $g(f(x))$.

$$f(g(x)) = (x-1)^2 + 2(x-1) = x^2 - 1$$
$$g(f(x)) = x^2 + 2x - 1$$

2. If $f(x) = 3x + 2$ and $g(x) = 4x + c$, find $f(g(x))$ and $g(f(x))$ and find a value of c for which $f(g(x)) = g(f(x))$.

$$f(g(x)) = 3(4x + c) + 2 = 12x + 3c + 2$$
$$g(f(x)) = 4(3x + 2) + c = 12x + 8 + c$$
$$\text{we must have } 3c + 2 = 8 + c \text{ so } c = 3$$

3. Find the range of the functions:

   (a) $y = x^2 + 3$, $-4 \leq x \leq 1$

$$x \in [-4, 1], \quad y \in [3, 19]$$

   (b) $y = 4 - 3x^2$, $-2 \leq x \leq 3$.

$$x \in [-2, 3], \quad y \in [-23, 4]$$

4. Find the inverse of the function $f(x) = 4x - 7$. Verify that $f(f^{-1}(x)) = x$ and that $f^{-1}(f(x)) = x$.

$$\text{If } y = 4x - 7 \text{ then we have } x = \frac{y+7}{4} \text{ so } f^{-1}(x) = \frac{x+7}{4}.$$
$$f(f^{-1}(x)) = 4\left(\frac{x+7}{4}\right) - 7 = x \qquad f^{-1}(f(x)) = \frac{(4x-7)+7}{4} = x$$

5. Find the inverse of the function $f(x) = \dfrac{2x-3}{x-4}, \quad x \neq 4$

$$\text{If } y = \frac{2x-3}{x-4}, \quad y(x-4) = 2x - 3, \quad xy - 4y = 2x - 3, \quad xy - 2x = 4y - 3,$$
$$x(y-2) = 4y - 3, \quad x = \frac{4y-3}{y-2} \text{ and thus } f^{-1}(x) = \frac{4x-3}{x-2}, \quad x \neq 2$$

6. Determine whether the following functions are odd, even or neither.
   (i) $x^4 + 3x^2 - 2$, (ii) $x^3 - x^2$, (iii) $x^3 - 3x$,
   (i) Even, e.g $f(1) = 1 + 3 - 2 = 2, \quad f(-1) = 1 + 3 - 2 = 2$
   (ii) neither e.g $f(1) = 0, \quad f(-1) = -2$
   (iii) odd e.g. $f(1) = (-2), \quad f(-1) = 2$

7. Sketch, on the same axes, the graphs of

   (a) x^2 and $x^2 - 4$ for $-4 \leq x \leq 4$.

   (b) x^2 and $(x+1)^2$ for $-4 \leq x \leq 4$.

8. Sketch the graph of (i) $3x - 5y - 30 = 0$, (ii) $5x + 3y - 15 = 0$ (iii) $y = |x - 1|$

9. Sketch the graph of (i) $y = 3e^x$, (ii) $y = x^2 - 3x + 2$

10. Find the equation of the straight line which passes through the points $(1, -2)$ and $(-2, 7)$. Find the points at which the line crosses the x-axis and the y-axis.

    Using the formula with $x_1 = 1, y_1 = -2, x_2 = -2$ and $y_2 = 7$ we have

$$y + 2 = \frac{-2 - 7}{1 + 2}(x - 1) \rightarrow y + 3x = 1$$

11. Find the centre and radius of the circle given by

$$x^2 + y^2 - 6x + 2y - 15 = 0.$$

    Complete the squares for both x and y

$$x^2 - 6x + 9 + y^2 + 2y + 1 - 9 - 1 - 15 = 0 \rightarrow (x - 3)^2 + (y + 1)^2 = 25$$

    The centre of the circle is at $(3, -1)$ and the radius is 5.

12. Use the rules of logarithms to simplify the following:
    (a) $\ln x + \ln x^2$, (b) $\ln e^3$, (c) $\ln(\frac{1}{2}) + \ln 4$,
    (d) $e^{5 \ln x}$, (e) $\ln 6 - \ln 3$, (f) $\ln(x + xy) - \ln x$.

    (a)$3\ln(x)$ (b) 3 (c) $\ln 2$ (d) x^5 (e) $\ln 2$ (f) $\ln(1 + y)$

More Challenging Questions

13. Find the domain and the range of the functions:

    (a) $f(x) = \sqrt{1 - x^2}$ (b) $g(x) = \sqrt{x^2 + x - 2}$ (c) $h(x) = \dfrac{1}{x^2 + x - 2}$.

    (a) Domain is $|x| \leq 1$ or $-1 \leq x \leq 1$ or $x \in [-1, 1]$. Range $f(x) \in [0, 1]$ (positive square root).
    (b) We must have $x^2 + x - 2 \geq 0$. Factorise $(x - 1)(x + 2)$ so we have either $x \leq -2$ or $x \geq 1$. Check by sketching the graph. The range is $f(x) \in [0, \infty)$.
    (c) We cannot have the denominator zero so the domain is $x \in \mathbb{R} \setminus \{-2, 1\}$ (All the real numbers except $x = -2$ or $x = 1$.) The range is $f(x) \in \mathbb{R} \setminus 0$.

14. Show that if $f(x)$ and $g(x)$ are both odd functions, then $h(x) = f(x) \times g(x)$ is an even function.

    $f(-x) = -f(x), \quad g(-x) = -g(x)$
    thus $\quad h(x) = f(x)g(x) = -f(-x) \times -g(-x) = f(-x)g(-x) = h(-x)$

15. Show that $f(x) = \dfrac{x+1}{x-1}$ is one-to-one. Find f^{-1} and show that your solution is correct. Can you explain the form of $f^{-1}(x)$?

    Let $\dfrac{x+1}{x-1} = \dfrac{y+1}{y-1}$. Then $(y - 1)(x + 1) = y + 1)(x - 1)$ and thus

    $xy - x - 1 + y = xy + x - y - 1$ so $y = x$ and the function is one to one

    $f^{-1}(x) = \dfrac{x+1}{x-1}$
    The function is its own inverse since its graph is symmetric about the line $x = y$.

    $$f^{-1}(f(x)) = \frac{\dfrac{x+1}{x-1} + 1}{\dfrac{x+1}{x-1} - 1} = x$$

16. Evaluate

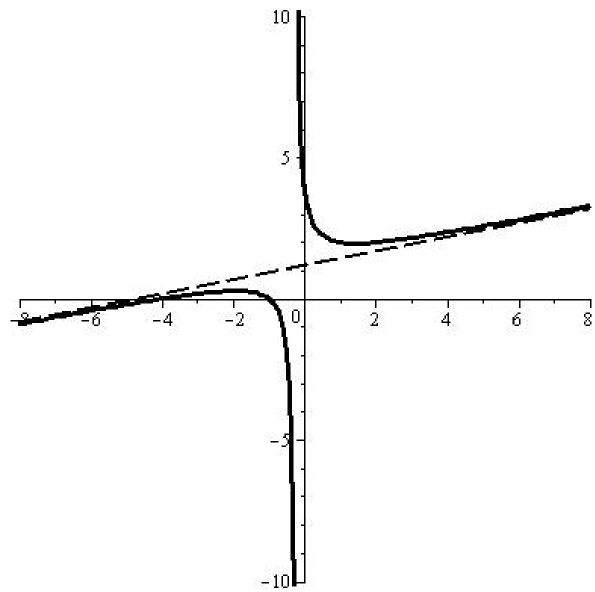
    $$(a) \quad \lim_{x \to \infty} \frac{2x+3}{x-1}, \quad (b) \quad \lim_{x \to 3} \frac{x-3}{x^2-9}, (c) \quad \lim_{x \to 1} \frac{x-1}{x^3-1}$$

    (a) 2
    (b) factorise the denominator and cancel the $(x - 1)$ factor, $\dfrac{1}{6}$
    (c) factorise the denominator $x^3 - 1 = (x - 1)(x^2 + x + 1)$ and cancel $(x - 1)$ as before.

17. Sketch the following graphs

$$(a) \quad y = \frac{x^2+5x+4}{4x+1}, \quad (b) \quad y = \frac{4x+1}{x^2+5x+4}$$

(a)



(b)