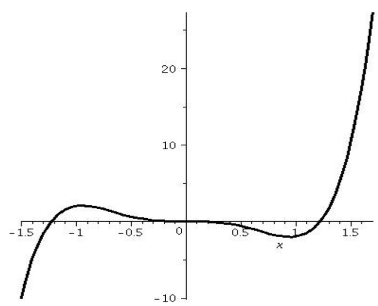
MAT 1015 Techniques in Calculus I Autumn 2009

Coursework 2 SOLUTIONS

1. Each of the following functions is defined on its maximal domain. In each case sketch a graph of the function, stating the equations of any asymptotes and giving the co-ordinates of any stationary points and any points of intersection with the axes.

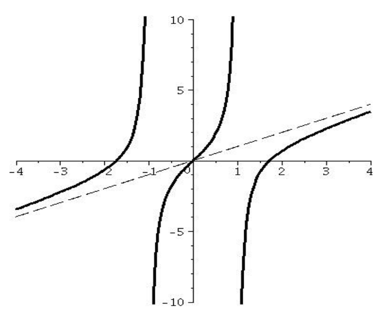
   (a) $f(x) = 4x^5 - 6x^3$, (b) $g(x) = \dfrac{3x - x^3}{1 - x^2}$, (c) $h(x) = \dfrac{x^2 - 3x + 3}{x^2 - 1}$.

   (a) When $f(x) = 0$ we have $x = 0$ and $x = \pm\sqrt{\dfrac{3}{2}}$. Also $f(0) = 0$. The extrema are where $f'(x) = 20x^4 - 18x^2 = 0$, $\quad x = 0$ and $x = \pm\dfrac{3}{\sqrt{10}}$ corresponding to $f(x) = \pm\frac{81}{125}\sqrt{10}$. As $x \to \infty, f(x) \to \infty$ and As $x \to -\infty, f(x) \to -\infty$



   (b) There are vertical asymptotes at $x = 1$ and $x = -1$. $g(0) = 0$. When $g(x) = 0 \quad x = 0, \pm\sqrt{3}$. For large x $g(x) \sim x$ by long division.

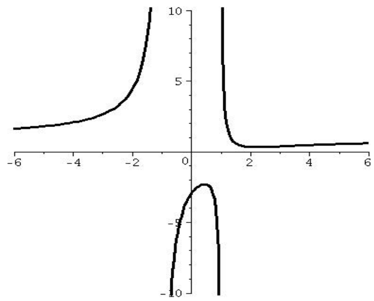
   Since $g'(x) = \dfrac{3 + x^4}{(x^2 - 1)^2} \neq 0$ for any x there are no stationary points.



   (c) There are vertical asymptotes at $x = 1$ and $x = -1$.

   $h'(x) = -\dfrac{-8\,x + 3 + 3\,x^2}{(-1 + x^2)^2}$ so there are stationary points at $x = \dfrac{4}{3} \pm \dfrac{1}{3}\sqrt{7}$.

   As $x \to \pm\infty \quad h(x) \to 1$



2. Use standard trigonometric identities to show that

$$\cos^2 x \sin x \equiv \frac{1}{2}\sin 2x \cos x \quad \text{and} \quad \cos A \sin B \equiv \frac{1}{2}\left(\sin(A+B) - \sin(A-B)\right),$$

hence express $\cos^2 x \sin x$ in terms of $\cos x$ and $\sin x$ and integrate this expression with respect to x.

(a) Also, integrate $\cos^2 x \sin x$ with respect to x using the substitution $u = \cos x$.

(b) Use your two integrals to deduce a formula for $\cos 3x$ in terms of $\cos x$.

$\cos x \cos x \sin x = \dfrac{1}{2}\cos x \sin 2x$

$\sin(A+B) = \sin A \cos B + \cos A \sin B$ and $\sin(A-B) = \sin A \cos B - \cos A \sin B$, hence, by subtraction we have

$\cos A \sin B \equiv \dfrac{1}{2}\left(\sin(A+B) - \sin(A-B)\right),$

Thus $\dfrac{1}{2}\cos x \sin 2x = \dfrac{1}{4}(\sin 3x + \sin x)$. Then

$$\int \frac{1}{4}(\sin 3x + \sin x)dx = -\frac{1}{4}\left(\frac{1}{3}\cos 3x + \cos x\right)$$

Also, directly

$$\int \sin x \cos^2 x dx = \frac{1}{3}\cos^3 x$$

Equating the two results we obtain

$$\cos 3x = 4\cos^3 x - 3\cos x$$

3. Solve the equation $3\sinh x + 4\cosh x = 3$.

$$3\sinh x + 4\cosh x = 3 \Rightarrow \frac{7e^x}{2} + \frac{e^{-x}}{2} = 3$$

Now let $e^x = y$ and multiply out to obtain $7y^2 - 6y + 1 = 0$ and solve to get $y = \dfrac{3 \pm \sqrt{2}}{7}$

Thus $x = \ln \dfrac{3 \pm \sqrt{2}}{7}$

4. (a) Express $\dfrac{1-2x}{(x+2)(x+4)^2}$ in partial fractions and hence find the exact value of $\displaystyle\int_{-1}^{1} \frac{1-2x}{(x+2)(x+4)^2}dx$.

$$\frac{1-2x}{(x+2)(x+4)^2} = \frac{5}{4(x+2)} - \frac{5}{4(x+4)} - \frac{9}{2(x+4)^2}$$

Hence

$$\int_{-1}^{1} \frac{1-2x}{(x+2)(x+4)^2}dx = \frac{5}{4}\ln|x+2| - \frac{5}{4}\ln|x+4| + \frac{9}{2(x+4)}\Bigg|_{-1}^{1} = -\frac{3}{5} + \frac{5}{2}\ln 3 - \frac{5}{4}\ln 5$$

(b) Show that $x+2$ is a factor of $x^3 - 3x + 2x^2 - 6$ and express $\dfrac{x(x^2-x+1)}{x^3-3x+2x^2-6}$ in partial fractions over (i) $\mathbb{Q}$ and (ii) $\mathbb{R}$.

$f(-2) = 0$ so $x = 2$ is a factor of the denominator which factorises as $(x+2)(x^2-3)$, then

$$\frac{x(x^2-x+1)}{x^3-3x+2x^2-6} = 1 + \frac{11x-18}{x^2-3} - \frac{14}{x+2}$$

by partial fractions over $\mathbb{Q}$ in the usual way, while over $\mathbb{R}$ we have

$$1 + \frac{11+6\sqrt{3}}{2(x+\sqrt{3})} - \frac{6\sqrt{3}-11}{2(x-\sqrt{3})} - \frac{14}{x+2}$$

5. The equation of a curve is given as $y = \dfrac{\sqrt{x}}{x+2}$ where $x > 0$.

(a) Find the gradient at the point where $x = 4$.

(b) Find and classify the stationary point.

(c) Find the volume of the solid generated by rotating the area bounded by the curve, the x-axis and the lines $x = 3$ and $x = 5$ around the x-axis.

(a)

$$y'(x) = -\frac{x-2}{2\sqrt{x}(x+2)^2} \Rightarrow y'(4) = -\frac{1}{72}$$

(b) $y' = 0$ when $x = 2, y = \dfrac{\sqrt{2}}{4}$

$$y'' = \frac{x^2-12x-4}{4x^{\frac{3}{2}}(x+2)^3}, \qquad y''(2) = -\frac{\sqrt{2}}{64} \Rightarrow \quad \text{maximum}$$

or y' goes from negative to positive as x goes from $x < 2$ to $x > 2$.

(c)

$$V = \int_3^5 \pi \left(\frac{\sqrt{x}}{x+2}\right)^2 dx = \pi \left(\frac{2}{x+2} + \ln|x+2|\right)\Bigg|_3^5 = \pi \left(\ln 7 - \ln 5 - \frac{4}{35}\right)$$

6. (a) Find a and b which satisfy the identity

$$e^{\left(\frac{a}{x+1}\right)} \times e^{\left(\frac{b}{x+2}\right)} = e^{\left(\frac{1}{x^2+3x+2}\right)}$$

Using partial fractions

$$\frac{1}{x^2+3x+2} = \frac{1}{x+1} - \frac{1}{x+2} \Rightarrow a = 1, b = -1$$

(b) Find the natural logarithm of

$$\frac{e^{\left(\frac{1}{x+1}\right)} \times e^{\left(\frac{1}{x}\right)}}{e^{\left(\frac{1}{x^2}\right)}}$$

$$\ln e^{\frac{2x^2-1}{x^2(x+1)}} = \frac{2x^2-1}{x^2(x+1)}$$

7. (a) Starting from the definition of the sinh function in terms of exponentials, derive a formula for arcsinh x in terms of natural logarithms.

$\sinh x = \frac{1}{2}(e^x - e^{-x}) = y$ Now solve for x in terms of y finding an inverse in the usual way.

$e^{2x} - 2ye^x - 1 = 0 \Rightarrow e^x = y + \sqrt{1+y^2}$ positive root only $\Rightarrow \sinh^{-1} x = \ln(x + \sqrt{1+x^2})$

(b) Given that $\sinh x = \cot\theta$, where $\theta \in \left(0, \frac{\pi}{2}\right)$, use this result to show that $x = \ln(\text{cosec } x + \cot\ x)$. Hence find $\frac{dx}{d\theta}$ in terms of θ.

If $\sinh x = \cot\theta$ then $x = \ln(\cot\theta + \sqrt{1+\cot^2\theta}) = \ln(\cot\theta + \text{cosec}\theta)$

then

$$\frac{dx}{d\theta} = \frac{-\text{cosec}^2\theta - \text{cosec}\theta\cot\theta}{\cot\theta + \text{cosec}\theta} = -\left(\frac{\frac{1}{\sin^2\theta} + \frac{\cos\theta}{\sin^2\theta}}{\frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}}\right)$$

$$\frac{dx}{d\theta} = -\left(\frac{1+\cos\theta}{\sin^2\theta}\right)\left(\frac{\sin\theta}{1+\cos\theta}\right) = -\text{cosec}\theta$$

8. Find, in their simplest forms

(a) $\frac{d}{dx}\,\mathrm{arctanh}(\sin x)$ (b) $\frac{d}{dx}\,\frac{1}{x-\sin x}$ (c) $\frac{d}{dx}\,\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}$.

(a)

$$\frac{d}{dx}\,\mathrm{arctanh}(\sin x) = \frac{\cos x}{1+\sin^2 x} = \sec x$$

(b)

$$\frac{d}{dx}\,\frac{1}{x-\sin x} = \frac{\cos x - 1}{(x-\sin x)^2}$$

(c)

$$\frac{d}{dx}\,\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} = 2\,\frac{x}{\left(\sqrt{1+x^2}+1\right)^2\sqrt{1+x^2}}$$

9. (a) Use logarithmic differentiation to find $\frac{dy}{dx}$ when

(i) x^{x^2} (ii) $y = 2^{x\tan x}$.

(i) $y = x^{x^2}$. Take logs $\ln y = x^2 \ln x$ and differentiate

$\frac{1}{y}y' = x + 2x\ln x$, hence $y' = x^{x^2}(x + 2x\ln x)$

(ii)$y = 2^{x\tan x}$, Take logs $\ln y = x\tan x \ln 2$ and differentiate $\frac{1}{y}y' = \ln 2(x\sec^2 x + \tan x)$

Thus $y' = 2^{x\tan x}\ln 2(x\sec^2 x + \tan x)$

(b) The equation of a curve is $x^3 + y = \ln x - y^3 + 3$. Find $\frac{dy}{dx}$ when $x = 1$.

$x^3 + y = \ln x - y^3 + 3$, differentiate w.r.t x to get $3x^2 + y' = \frac{1}{x} - 3y^2y'$.

Now rearrange

$$y' = \frac{1-3x^3}{x(1+3y^2)}$$

If $x = 1$ then we have $y^3 + y - 2 = 0 \Rightarrow (y-1)(y^2+y+2) = 0$. The only real root is $y = 1$. Hence $y'(1) = -\frac{1}{2}$

(c) The parametric equations of a curve are $x = 2a\,\mathrm{sech}^3 t$, $y = 3a\tanh^2 t$, where a is a positive constant. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

$\dot{x} = -6a\,\mathrm{sech}^3 t\tanh t$, $\dot{y} = 6a\,\mathrm{sech}^2 t\tanh t$.

thus

$$\frac{dy}{dx} = \frac{6a\,\mathrm{sech}^2 t\tanh t}{-6a\,\mathrm{sech}^3 t\tanh t} = -\cosh t$$

therefore
$$\frac{d^2y}{dx^2} = -\sinh t\frac{dt}{dx} = \frac{-\sinh t}{-6a\text{sech}^3 t\tanh t} = \frac{\cosh^4 t}{6a}$$

(d) Use Leibniz Rule to find the fourth derivative of $x^6 \sin 3x$ with respect to x.

$$y^{[4]} = \sum_{r=0}^{4} \binom{4}{r} (\sin 3x)^{[4-r]} (x^6)^{[r]}$$
$$= 81x^6 \sin 3x - 648x^5 \cos 3x - 1620x^4 \sin 3x + 1440x^3 \cos 3x + 360x^2 \sin 3x$$

10. Find the equation of the straight line of minimum slope which is tangent to the curve
$$y = 3 + 6x - 2x^2 + \frac{x^3}{2} + \frac{x^4}{12}.$$
$$y' = \frac{x^3}{3} + \frac{3x^2}{2} - 4x + 6, \qquad y'' = x^2 + 3x - 4, \qquad y''' = 2x + 3$$
The gradient will be least when y' has a minimum, when $y'' = 0$, ie. $x^2 + 5x - 4 = 0 \Rightarrow x = 1$ and -4. Now $y'''(1) = 5 > 0$ while $y'''(-4) = -5 < 0$. Thus the minimum occurs at $x = 1, y' = \frac{23}{6}$ and $y = \frac{91}{12}$. The equation of the tangent at this point is
$$y - \frac{91}{12} = \frac{23}{6}(x-1) \Rightarrow 6y = 23x + \frac{45}{2}$$

11. Prove De Moivre's Theorem by induction for all integers, $n \geq 1$

(a) Expand $(\cos\theta + i\sin\theta)^5$ by the binomial theorem.
$$(\cos\theta + i\sin\theta)^5 = (\cos(\theta))^5 + 5\,i\,(\cos(\theta))^4 \sin(\theta) - 10\,(\cos(\theta))^3 (\sin(\theta))^2$$
$$-10\,i\,(\cos(\theta))^2 (\sin(\theta))^3 + 5\,\cos(\theta)(\sin(\theta))^4 + i\,(\sin(\theta))^5$$

(b) Hence express $\cos 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$
$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

(c) Expand $\left(z + \frac{1}{z}\right)^4$ by the binomial theorem.
$$z^4 + 4\,z^2 + 6 + 4\,z^{-2} + z^{-4} = \left(z^4 + \frac{1}{z^4}\right) + 4\left(z^2 + \frac{1}{z^2}\right) + 6$$

(d) Hence express $\cos^4\theta$ in terms of cosines of multiples of θ. If $z = e^{i\theta}$ then $\cos\theta = 2\left(z + \frac{1}{z}\right)$ so that
$$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6 \Rightarrow \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$$

(e) Use this result to find $\int_0^{\frac{\pi}{4}} \cos^4 \theta d\theta$.

$$\int_0^{\frac{\pi}{4}} \cos^4 \theta d\theta = \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3\theta}{8}\Big|_0^{\frac{\pi}{4}} = \frac{1}{4} + \frac{3\pi}{32}$$

12. * Eliminate the parameter from the parametric equation

$$x = \frac{3t}{1+t^3} \quad y = \frac{3t^2}{1+t^3} \quad (t \neq -1)$$

to obtain an equation in x and y. Sketch the curve.

$$x^3 = \left(\frac{3t}{1+t^3}\right)^3 \qquad y^3 = \left(\frac{3t^2}{1+t^3}\right)^3$$

$$x^3 + y^3 = \frac{27t^2(1+t^3)}{(1+t^3)^3} = \frac{27t^3}{(1+t^3)^2} = 3 \times \frac{3t}{1+t^3} \times \frac{3t^2}{1+t^3} = 3xy$$