MAT 1015 Techniques in Calculus I Autumn 2009

Coursework 2

Please hand in your solution to me at or before the 10:00 lecture on Friday 27th November, clearly marked with **your name and that of your tutor**. You should show all your working as part of your answers. The question with an asterisk (*) may be a little more challenging.

1. Each of the following functions is defined on its maximal domain. In each case sketch a graph of the function, stating the equations of any asymptotes and giving the co-ordinates of any stationary points and any points of intersection with the axes.

   (a) $f(x) = 4x^5 - 6x^3$, (b) $g(x) = \dfrac{3x - x^3}{1 - x^2}$, (c) $h(x) = \dfrac{x^2 - 3x + 3}{x^2 - 1}$.

2. Use standard trigonometric identities to show that

   $$\cos^2 x \sin x \equiv \frac{1}{2} \sin 2x \cos x \quad \text{and} \quad \cos A \sin B \equiv \frac{1}{2} \left(\sin(A + B) - \sin(A - B)\right),$$

   hence express $\cos^2 x \sin x$ in terms of $\cos x$ and $\sin x$ and integrate this expression with respect to x.

   (a) Also, integrate $\cos^2 x \sin x$ with respect to x using the substitution $u = \cos x$.

   (b) Use your two integrals to deduce a formula for $\cos 3x$ in terms of $\cos x$.

3. Solve the equation $3 \sinh x + 4 \cosh x = 3$.

4. (a) Express $\dfrac{1 - 2x}{(x + 2)(x + 4)^2}$ in partial fractions and hence find the exact value of $\displaystyle\int_{-1}^{1} \frac{1 - 2x}{(x + 2)(x + 4)^2} dx$.

   (b) Show that $x + 2$ is a factor of $x^3 - 3x + 2x^2 - 6$ and express $\dfrac{x(x^2 - x + 1)}{x^3 - 3x + 2x^2 - 6}$ in partial fractions over (i) $\mathbb{Q}$ and (ii) $\mathbb{R}$.

5. The equation of a curve is given as $y = \dfrac{\sqrt{x}}{x + 2}$ where $x > 0$.

   (a) Find the gradient at the point where $x = 4$.

   (b) Find and classify the stationary point.

   (c) Find the volume of the solid generated by rotating the area bounded by the curve, the x-axis and the lines $x = 3$ and $x = 5$ around the x-axis.

6. (a) Find a and b which satisfy the identity

   $$e^{\left(\frac{a}{x+1}\right)} \times e^{\left(\frac{b}{x+2}\right)} = e^{\left(\frac{1}{x^2+3x+2}\right)}$$

(b) Find the natural logarithm of

$$\frac{e^{\left(\frac{1}{x+1}\right)} \times e^{\left(\frac{1}{x}\right)}}{e^{\left(\frac{1}{x^2}\right)}}$$

7. (a) Starting from the definition of the sinh function in terms of exponentials, derive a formula for arcsinh x in terms of natural logarithms.

   (b) Given that $\sinh x = \cot\theta$, where $\theta \in \left(0, \frac{\pi}{2}\right)$, use this result to show that $x = \ln(\text{cosec } x + \cot\ x)$. Hence find $\frac{dx}{d\theta}$ in terms of θ.

8. Find, in their simplest forms

   (a) $\frac{d}{dx}\,\text{arctanh}(\sin x)$ (b) $\frac{d}{dx}\,\frac{1}{x - \sin x}$ (c) $\frac{d}{dx}\,\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1}$.

9. (a) Use logarithmic differentiation to find $\frac{dy}{dx}$ when

   (i) x^{x^2} (ii) $y = 2^{x\tan x}$.

   (b) The equation of a curve is $x^3 + y = \ln x - y^3 + 3$. Find $\frac{dy}{dx}$ when $x = 1$.

   (c) The parametric equations of a curve are $x = 2a\text{sech}^3 t$, $y = 3a\tanh^2 t$, where a is a positive constant. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of t.

   (d) Use Leibniz Rule to find the fourth derivative of $x^6 \sin 3x$ with respect to x.

10. Find the equation of the straight line of minimum slope which is tangent to the curve

$$y = 3 + 6x - 2x^2 + \frac{x^3}{2} + \frac{x^4}{12}.$$

11. Prove De Moivre's Theorem by induction for all integers, $n \geq 1$

   (a) Expand $(\cos\theta + i\sin\theta)^5$ by the binomial theorem.

   (b) Hence express $\cos 5\theta$ in terms of powers of $\sin\theta$ and $\cos\theta$

   (c) Expand $\left(z + \frac{1}{z}\right)^4$ by the binomial theorem.

   (d) Hence express $\cos^4\theta$ in terms of cosines of multiples of θ.

   (e) Use this result to find $\int_0^{\frac{\pi}{4}} \cos^4\theta \mathrm{d}\theta$.

12. * Eliminate the parameter from the parametric equation

$$x = \frac{3t}{1+t^3} \quad y = \frac{3t^2}{1+t^3} \quad (t \neq -1)$$

to obtain an equation in x and y. Sketch the curve.