UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Computing

Level HE2 Examination

Module MAT 2016 Computational Operations Research

SOLUTIONS

Time allowed – 2 hrs Autumn Semester 2008/09

Attempt THREE questions.

If a candidate attempts more than THREE questions
only the best THREE questions will be taken into account.

Question 1

A refinery produces two types of lubricating oil, type A and type B. To produce 1000 litres of type A oil requires two hours in process 1, two hours in process 2 and six hours in process 3. To produce 1000 litres of type B oil requires four hours in process 1, two hours in process 2 and three hours in process 3. The availability of the process 1 plant is twelve hours per day, that of process 2 is seven hours per day and that of process 3 twenty hours per day. The profit on 1000 litres of Type A is £40 and on Type B £50.

The company wants to know how much of each type of oil to produce in order to maximise their profit per day, assuming that all the production can be sold.

(a) What are the decision variables for this problem?
x_1=***number of litres of type A oil produced daily (in*** 1000***'s)***
x_2=***number of litres of type B oil produced daily (in*** 1000***'s)*** [2]

(b) What is the objective function?
P=***profit per day in £***
$P = 40x_1 + 50x_2$ [2]

(c) State the constraints as inequalities.
$2x_1 + 4x_2 \leq 12$ ***(process 1)***
$2x_1 + 2x_2 \leq 7$ ***(process 2)***
$6x_1 + 3x_2 \leq 20$ ***(process 3)*** [3]

(d) Sketch the feasible region, indicating the coordinates of each of the corner points. On your diagram, show the contour associated with a daily profit of £200. [4]

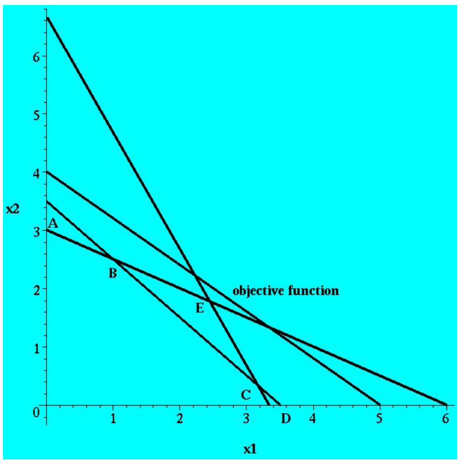

Figure 1: feasible region

(e) Using this contour or otherwise (but NOT by using the Simplex Method), determine the optimum solution to the problem, justifying your approach. [4]

EITHER
the coordinates of the four corner points of the feasible region and the profit level associated with them are as follows:

$A = (0, 3)$, $P(0, 3) = 150$
$B = (1, \frac{5}{2})$, $P(1, \frac{5}{2}) = 165$ ***(solution of*** $2x_1 + 4x_2 = 12$ ***and*** $2x_1 + 2x_2 = 7$***)***
$C = (\frac{19}{6}, \frac{1}{3})$, $P(\frac{19}{6}, \frac{1}{3}) = \frac{430}{3}$ ***(solution of*** $2x_1 + 2x_2 = 7$ ***and*** $6x_1 + 3x_2 = 20$***)***
$D = (\frac{10}{3}, 0)$, $P(\frac{10}{3}, 0) = \frac{400}{3}$

OR
By placing a straight edge parallel to the contour line and moving it away from the origin we can establish that corner B is the last point of the feasible region that the contour line will touch as the profit is increased. The coordinates of this point are $(1, \frac{5}{2})$ ***and the value of the objective function is*** 165.

(f) How much of each type of oil should the Company produce each day? [2]
The Company should produce 1000 ***litres of Type A oil and*** 2,500 ***litres of Type B oil each day, resulting in a profit of*** £165 ***per day.***

(g) The demand for Type A oil has risen greatly and the Company intends to increase its price. By how much can the profit on Type A oil be increased before the Company should produce this type of oil only? How much profit would the company make at this profit level. [3]
The optimal solution in this case will be at point D. The absolute value of the slope of the objective function must exceed that of the constraint line $6x_1 + 3x_2 = 20$***. If the profit on Type A is*** p ***we must have*** $\frac{p}{50} > 2$ ***so we need*** $p > 100$. ***At a profit of*** £100 ***per*** 1000 ***litres of Type A produced the Company will make*** $\frac{10}{3} \times 100 = £333\frac{1}{3}$ ***per day profit.***

(h) By reorganising the workforce it is possible to increase the number of hours that the plant for process 2 is available. In order to maximise profit, how many extra hours should process 2 be made available? (carefully explain your reasoning) How much of each type of oil will the refinery then produce and what will be the company's profit. [5]
The amount of time available for process 2 can be increased until the line representing that constraint passes though point E at which point all the available time for process 1 and process 2 will have been fully utilised. The co-ordinates of E are $(\frac{22}{9}, \frac{16}{9})$***, substituting these values into the expression for the constraint of process 2 gives*** $2 \times \frac{22}{9} + 2 \times \frac{16}{9} = 8\frac{4}{9}$. ***Thus an extra*** $1\frac{5}{9}$ ***hours of process 2 should be made available. The Company should produce*** $2,444\frac{4}{9}$ ***litres of Type A oil and*** $1,777\frac{7}{9}$ ***of Type B oil and the daily profit will be*** $£186\frac{2}{3}$.

Question 2

(a) Put the following problem into standard form. [Do NOT attempt to solve it]
Minimize

$$z = 4x_1 - 7x_2 + 3x_3$$

subject to the constraints

$$\begin{aligned} 3x_1 - 4x_2 + x_3 &\geq 12 \\ x_1 + 2x_2 - 3x_3 &\leq -4 \\ x_1 + x_2 + x_3 &= 2 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

Objective function, maximise

$$z' = -4x_1 + 7x_2 - 3x_3.$$

[1]

Add and subtract slack variables for each constraint

$$3x_1 - 4x_2 + x_3 - x_4 = 12$$

$$x_1 + 2x_2 - 3x_3 + x_5 = -4 \Longrightarrow -x_1 - 2x_2 + 3x_3 - x_5 = 4.$$

[2]

Solve for one variable* (could also be x_1 or x_2)

$$x_3 = 2 - x_1 - x_2.$$

[1]

Substitute for x_3 and simplify

$$z' = -4x_1 + 7x_2 - 3(2 - x_1 - x_2) \Longrightarrow z' = -x_1 + 10x_2 - 6$$

$$3x_1 - 4x_2 + (2 - x_1 - x_2) - x_4 = 12 \Longrightarrow 2x_1 - 5x_2 - x_4 = 10$$

$$-x_1 - 2x_2 + 3(2 - x_1 - x_2) - x_5 = 4 \Longrightarrow -4x_1 - 5x_2 - x_5 = -2 \Longrightarrow 4x_1 + 5x_2 + x_5 = 2.$$

[4]

Ensure that $x_3 \geq 0$

$$x_1 + x_2 \leq 2 \Longrightarrow x_1 + x_2 - x_6 = 2$$

[1]

***ALSO ACCEPT**
Replace $x_1 + x - 2 + x_3 = 2$ by

$$x_1 + x - 2 + x_3 \leq 2 \Longrightarrow x_1 + x_2 + x_3 + x_6 = 2$$

and

$$x_1 + x - 2 + x_3 \geq 2 \Longrightarrow x_1 + x_2 + x_3 - x_7 = 2$$

[6]

(b) Consider the following problem:

Maximize

$$z = 4x_1 + 5x_2$$

subject to the constraints

$$5x_1 + 2x_2 \leq 12$$
$$3x_1 + 4x_2 \leq 10$$

and

$$x_1 \geq 0, \quad x_2 \geq 0$$

(i) Write this problem in standard form. [2]

Maximize $z = 4x_1 + 5x_2$

subject to

$$5x_1 + 2x_2 + x_3 = 12$$
$$3x_1 + 4x_2 + x_4 = 10$$
$$x_i \geq 0, \quad i = 1 \ldots 4$$

(ii) Find an initial basic feasible solution, stating which are the basic variables and which are the non-basic variables. [2]
x_1 ***and*** x_2 ***are both zero, they are the non-basic variables.*** $x_3 = 12$ ***and*** $x_4 = 10$ ***are the basic variables.***

(iii) Construct the initial tableau for the Simplex Method. [2]

Basic	z	x_1	x_2	x_3	x_4	Solution
x_3	0	5	2	1	0	12
x_4	0	3	4	0	1	10
z	1	-4	-5	0	0	0

(iv) Use the tableau form of the Simplex Method to find the optimal solution of the problem. At each stage, state the entering and departing variables from the basis and explain your reasoning. [9]

The largest negative value in the z ***row is*** -5 ***so*** x_2 ***is the entering variable. The lowest row quotient is*** $\frac{10}{4} = 2\frac{1}{2}$ ***so*** x_4 ***is the departing variable.***

Basic	z	x_1	x_2	x_3	x_4	Solution	
x_3	0	$\frac{7}{2}$	0	1	$-\frac{1}{2}$	7	$r_1 := r_1 - 2r_2$
x_2	0	$\frac{3}{4}$	1	0	$\frac{1}{4}$	$\frac{5}{2}$	$r_2 := \frac{r_2}{4}$
z	1	$-\frac{1}{4}$	0	0	$\frac{5}{4}$	$\frac{25}{2}$	$r_3 := r_3 + 5r_2$

The largest negative value in the z row is now $-\frac{1}{4}$ so x_1 is the entering variable. The lowest row quotient is $\frac{7}{\frac{7}{2}} = 2$ so x_3 is the departing variable.

Basic	z	x_1	x_2	x_3	x_4	Solution	
x_1	0	1	0	$\frac{2}{7}$	$-\frac{1}{7}$	2	$r_1 := \frac{r_1}{\frac{7}{2}}$
x_2	0	0	1	$-\frac{3}{14}$	$\frac{5}{14}$	1	$r_2 := r_2 - \frac{3}{4}r_1$
z	1	0	0	$\frac{1}{14}$	$\frac{17}{14}$	13	$r_3 := r_3 + \frac{1}{4}r_1$

(v) From the optimal tableau, determine the optimal value of z and state the values of all the variables at this solution. [1]

The optimal values are $z = 13, \quad x_1 = 2, \quad x_2 = 1, \quad x_3 = 0, \quad x_4 = 0$

Question 3

(a) Consider the linear programming problem

maximise profit, given by $z = 3x_1 + 2x_2$

subject to these constraints

$x_1 + x_2 \leq 6$ [i]
$2x_1 + 3x_2 \leq 12$ [ii]
$3x_1 + x_2 \leq 12$ [iii]
$x_1 \geq 0, \quad x_2 \geq 0.$

The optimum simplex tableau for this problem is as follows:

Basic	z	x_1	x_2	x_3	x_4	x_5	Solution
x_3	0	0	0	1	$-\frac{2}{3}$	$-\frac{1}{7}$	$\frac{6}{7}$
x_2	0	0	1	0	$\frac{3}{7}$	$-\frac{2}{7}$	$1\frac{5}{7}$
x_1	0	1	0	0	$-\frac{1}{7}$	$\frac{3}{7}$	$3\frac{5}{7}$
z	1	0	0	0	$\frac{3}{7}$	$\frac{5}{7}$	$13\frac{5}{7}$

An additional variable x_6 is now to be added to the problem, requiring 1 unit of the resources represented by constraint (i), 2 units of the resources represented by constraint (ii) and 1 unit of the resources represented by constraint (iii) per unit of x_6. The profit per unit of x_6 is p.

(i) Express the new problem in terms of the original problem (HINT use variables x'_3, x'_4, x'_5, z').

maximise
$z' = 3x_1 + 2x_2 + px_6 \Longrightarrow z' = z + px_6$

subject to
$x_1 + x_2 + x_6 \leq 6 \Longrightarrow x_1 + x_2 + x_6 + x'_3 = 6 \Longrightarrow x'_3 = x_3 - x_6$
$2x_1 + 3x_2 + 2x_6 \leq 12 \Longrightarrow 2x_1 + 3x_2 + 2x_6 + x'_4 = 12 \Longrightarrow x'_4 = x_4 - 2x_6$
$3x_1 + x_2 + x_6 \leq 12 \Longrightarrow 3x_1 + x_2 + x_6 + x'_5 = 12 \Longrightarrow x'_5 = x_5 - x_6$

and
$x_i \geq 0, i = 3, 4, 5, 6, \quad x'_k \geq 0, k = 3, 4, 5$

[5]

(ii) How large can p be before the original solution ceases to be optimal?

Substituting for x_3, x_4 ***and*** x_5 ***in the equation for*** z ***represented by the***

SEE NEXT PAGE

original optimal tableau

$$\begin{aligned} z &= 13\tfrac{5}{7} - \tfrac{3}{7}x_4 - \tfrac{5}{7}x_5 \\ &= 13\tfrac{5}{7} - \tfrac{3}{7}(x_4' + 2x_6) - \tfrac{5}{7}(x_5' + x_6) \\ &= 13\tfrac{5}{7} - \tfrac{3}{7}x_4' - \tfrac{5}{7}x_5' - \tfrac{11}{7}x_6 \end{aligned}$$

thus

$$z' = z + px_6 = 13\tfrac{5}{7} - \tfrac{3}{7}x_4' - \tfrac{5}{7}x_5' + (p - \tfrac{11}{7})x_6$$

The coefficient of x_6 will be non-positive and thus z' will be optimal provided that $p \leq \frac{11}{7}$. [3]

(iii) Suppose that $p = 3$. Modify the original simplex tableau to take account of the new problem.

We substitute for x_3, x_4, x_5 in the equations represented by the original optimal tableau using the equations derived in the previous part of the question.

for x_3

$$x_3 - \tfrac{2}{3}x_4 - \tfrac{1}{7}x_5 = \tfrac{6}{7}$$
$$\implies (x_3' + x_6) - \tfrac{2}{3}(x_4' + 2x_6) - \tfrac{1}{7}(x_5' + x_6) = \tfrac{6}{7}$$
$$\implies x_3' - \tfrac{2}{3}x_4' - \tfrac{1}{7}x_5' - \tfrac{11}{21}x_6 = \tfrac{6}{7}$$

for x_2

$$x_2 + \tfrac{3}{7}x_4 - \tfrac{2}{7}x_5 = \tfrac{12}{7}$$
$$\implies x_2 + \tfrac{3}{7}(x_4' + 2x_6) - \tfrac{2}{7}(x_5' + x_6) = \tfrac{12}{7}$$
$$\implies x_2 + \tfrac{3}{7}x_4' - \tfrac{2}{7}x_5' + \tfrac{4}{7}x_6 = \tfrac{12}{7}$$

for x_1

$$x_1 - \tfrac{1}{7}x_4 + \tfrac{3}{7}x_5 = \tfrac{24}{7}$$
$$\implies x_1 - \tfrac{1}{7}(x_4' + 2x_6) + \tfrac{3}{7}(x_5' + x_6) + \tfrac{1}{7}x_6 = \tfrac{24}{7}$$

$$\implies x_1 - \tfrac{1}{7}x_4' + \tfrac{3}{7}x_5' + \tfrac{2}{7}x_6' = \tfrac{24}{7}$$

for z' substitute $p = 3$ from the previous section

$$z' = 13\tfrac{5}{7} - \tfrac{3}{7}x_4' - \tfrac{5}{7}x_5' + \tfrac{10}{7}x_6$$

The modified tableau is now

Basic	z	x_1	x_2	x_3'	x_4'	x_5'	x_6	Solution
x_3	0	0	0	1	$-\frac{2}{3}$	$-\frac{1}{7}$	$-\frac{11}{21}$	$\frac{6}{7}$
x_2	0	0	1	0	$\frac{3}{7}$	$-\frac{2}{7}$	$\frac{4}{7}$	$1\frac{5}{7}$
x_1	0	1	0	0	$-\frac{1}{7}$	$\frac{3}{7}$	$\frac{2}{7}$	$3\frac{5}{7}$
z	1	0	0	0	$\frac{3}{7}$	$\frac{5}{7}$	$-\frac{10}{7}$	$13\frac{5}{7}$

[5]

(iv) Determine the entering and departing variables for this new tableau and give your reasoning.

The only negative coefficient in the z' row is that in x_6 column, so x_6 is the entering variable. The only positive entry in the x_6 column corresponds to x_2 which is thus the departing variable. [2]

(v) Find the new optimal value of z, assuming that it will be obtained from a single iteration of the Simplex method on the new tableau. [You do NOT have to find the complete optimal tableau.]

The solution column value for x_6 is $\frac{12}{7} \times \frac{7}{4} = 3$. To eliminate the $\frac{10}{7}$ entry in the z' row requires the addition to row 4 of $\frac{10}{7}\times$ row 3. Thus $z' = 13\frac{5}{7} + \frac{10}{7} \times 3 = 18$. [2]

(b) The demand for paper at Surrey Printing Works is 800 boxes per day. The print shop manager is able to buy paper for £2 per box for orders of less than 2000 boxes and £1.50 per box for orders of 2000 boxes or more. Every time an order is placed, a fixed cost of £150 is incurred. The daily cost of storing paper is £0.05 per box. Determine the optimal order quantity and the associated cost per day.

β is the daily consumption of paper, h is the daily storage cost, c_1 and c_2 the basic and discount prices respectively, K is the set-up cost and q is the price break.

In this case $\beta = 800, \quad h = 0.05, \quad c_1 = 2, \quad c_2 = 1.5, \quad q = 2000, \quad K = 150.$
We find the EOQ with the formula

$$Q_m = \sqrt{\frac{2K\beta}{h}} = \sqrt{\frac{2 \times 150 \times 800}{0.05}} = 2190.89.$$

[1]

We round Q_m to 2191. ***Thus $q > Q_m$ so the optimal quantity is not in Zone 1.*** [1]

We find q_1 by solving $T_1(Q_m) = T_2(q_1)$ where

$$T_1(Q_m) = \beta c_1 + \frac{K\beta}{Q_m} + \frac{hQ_m}{2}$$

$$T_2(q_1) = \beta c_2 + \frac{K\beta}{q_1} + \frac{hq_1}{2}$$

i.e. [1]

$$800 \times 2 + \frac{150 \times 800}{2191} + \frac{0.05 \times 2191}{2} = 800 \times 1.5 + \frac{150 \times 800}{q_1} + \frac{0.05q_1}{2}$$

which simplifies to

$$0.25q_1^2 - 5,095.44q_1 + 1,200,000 = 0$$

[1]

$q_1 = 238$ ***or*** $24,568$ ***rounded to the nearest whole number***

[1]

We need the larger value of q_1 ***to find the boundary of Zone 2.*** $q < q_1$ ***so*** $Q_0^\star = q$. [1]

The optimal order quantity is 5000 ***boxes.*** [1]
The total cost is thus $T_2(5000) = £1349$ [1]

Question 4

(a) Set out each of the following transportation problems in the form of a tableau, having made the necessary additions to ensure that the problems are balanced. Carefully explain your reasoning.

(i) Factories F_1 and F_2 produce 18 and 15 units per day respectively. The daily demands of Warehouses W_1, W_2 and W_3 are 8,12 and 16 units per day.

Total supply is 33, total demand 36. We thus introduce a dummy factory with an output of 3 units per day and assign a transportation cost of a very large value P to ensure that no product is actually shipped from the dummy factory. The tableau is

	Warehouse 1	***Warehouse 2***	***Warehouse 3***	***Supply***
Factory 1	c_{11}	c_{12}	c_{13}	***18***
Factory 2	c_{21}	c_{22}	c_{23}	***15***
Dummy	***P***	***P***	***P***	***3***
Demand	***8***	***12***	***16***	

[3]

(ii) Factories F_1 and F_2 produce 25 and 18 units per day respectively. The daily demands of Warehouses W_1, W_2 and W_3 are 9,13 and 17 units per day.

Total supply is 43, total demand is 39. We thus introduce a dummy warehouse with a demand of 4 units per day. Since no product is shipped to the dummy warehouse we assign a zero transportation cost the shipments to the dummy. The tableau is

	Warehouse 1	***Warehouse 2***	***Warehouse 3***	***dummy***	***Supply***
Factory 1	c_{11}	c_{12}	c_{13}	***0***	***25***
Factory 2	c_{21}	c_{22}	c_{23}	***0***	***18***
Demand	***9***	***13***	***17***	***4***	

[3]

(b) Four warehouses are supplied by three factories. The supply available from each factory, the demand at each warehouse and the cost per unit of transporting goods from the factories to the warehouses are summarised in the following table:

	Warehouse 1	Warehouse 2	Warehouse 3	Warehouse 4	Supply
Factory 1	15	7	20	6	15
Factory 2	8	12	9	12	20
Factory 3	10	8	13	16	25
Demand	13	14	21	12	

(i) Use the Least-Cost method to find an initial basic feasible solution of this problem. [Do NOT use the North-West Corner method.]

The initial basic feasible solution is as follows:

	W_1	W_2	W_3	W_4	***Supply***
F_1	15	3 7	20	12 6	***15***
F_2	13 8	12	7 9	12	***20***
F_3	10	11 8	14 3	16	***25***
Demand	***13***	***11***	***21***	***12***	

[5]

(ii) Find the optimal solution of this problem, i.e. the solution that minimises the transportation costs, clearly showing and explaining your working.

We suppose that the cost of transporting 1 unit from factory i ***to warehouse*** j***,*** c_{ij} ***is made up of a dispatch cost*** λ_i ***and a reception cost*** μ_j ***such that***

$$c_{ij} = \lambda_i + \mu_j.$$

Since there are $n+m$ ***of the*** λ ***and*** μ ***variables but only*** $n+m-1$ ***basic variables we can arbitrarily assign a value to one of the*** λ_i***. We put*** $\lambda_1 = 0$ ***and compute the other values for*** λ_i ***and*** μ_j***.***

The shadow cost is the difference for each cell between the value of c_{ij}***, computed in this manner and the transportation cost. All of these shadow prices must be positive for an optimal solution.***

The largest negative shadow price is that of cell $(3,1)$***. Assigning*** ϵ ***units to this cell and making the concomitant adjustments to the other cells shows that*** $\epsilon \leq 13$***.***

	11	***7***	***12***	***6***	***Supply***
0	4 15	3 7	8 20	12 6	***15***
-3	$13-\epsilon$ 8	8 12	$7+\epsilon$ 9	9 12	***20***
1	ϵ -2 10	11 8	$14-\epsilon$ 3	9 16	***25***
Demand	***13***	***11***	***21***	***12***	

With this value of ϵ ***the final tableau has no negative shadow costs and is thus optimal:***

	9	***7***	***12***	***6***	***Supply***
0	6 15	3 7	8 20	12 6	***15***
-3	2 8	8 12	20 9	9 12	***20***
1	13 10	11 8	1 3	9 16	***25***
Demand	***13***	***11***	***21***	***12***	

[12]

(iii) What is the minimum total transportation cost?

The total cost is $13 \times 10 + 3 \times 7 + 11 \times 8 + 20 \times 9 + 1 \times 13 + 12 \times 6 = 504.$ [2]

INTERNAL EXAMINER: JOHN RAYMAN
EXTERNAL EXAMINER: T. FEARN