COM Mathematical Methods for Computing 2009

COURSEWORK II SOLUTIONS

1.
$$f(x) = x^2 + 4x + 3 = 0 \Rightarrow (x+1)(x+3) = 0 \Rightarrow x = (-3, -1)$$

(a) roots of $f(x-a) = 0$ are $(a-1, a-3)$

(b) roots of $f(3x) = 0$ are $(-1, -\frac{1}{3})$

(c) roots of $f\left(\frac{x}{2}\right) = 0$ are $(-2, -6)$

2. (a)
$$(2\sin^2 x - 3)(2\sin x - 1) = 0$$

whence $\sin x = \pm\sqrt{\dfrac{3}{2}}$ This solution we discard since $|\sin x| \leq 1$

and $\sin x = \dfrac{1}{2} \Rightarrow x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

(b)

(c)
$$(x^3+1)(x^3+8) = 0 \Rightarrow x^3 = -1, x = -1 \quad \text{and} \quad x^3 = -8, x = -2$$

(d)
$$e^{3x} - 7e^x + 6 = 0 \Rightarrow u^3 - 7u + 6 = 0$$

then $(u-1)(u-2)(u+3) = 0$ so $u = 1, 2, -3$ Hence $e^x = 1, x = 0$ and $e^x = 2, x = \ln 2$. Note that e^x is always positive so -3 is not a solution.

3. (a) (i) $(1, 6, 2)$ (ii) $(3, -3, -11)$ (iii) $(2, 3, -13)$

(b) (i) $\sqrt{83}$ (ii)$\sqrt{182}$

(c) (i) -34 (ii) -45

4. $$M = \begin{pmatrix} 2 & 3 \\ -1 & 6 \end{pmatrix}$$

characteristic polynomial $\lambda^2 - 8\lambda + 15 = 0,\qquad \lambda = 3, 5$
eigenvectors: for $\lambda = 5$: $(1, 1)$ for $\lambda = 3$: $(3, 1)$

$$M^{-1} = \frac{1}{15}\begin{pmatrix} 6 & -3 \\ 1 & 2 \end{pmatrix}$$

characteristic polynomial $\lambda^2 - \frac{8}{15}\lambda + \frac{1}{15} = 0,\qquad \lambda = \frac{1}{3}, \frac{1}{5}$
eigenvectors $(1, 1)$ and $(3, 1)$ as before
The eigenvalues of the inverse are the reciprocals of the eigenvalues of the matrix, the eigenvectors are the same. This is why:

$$M\mathbf{v} = \lambda\mathbf{v} \qquad \mathbf{v} = \lambda M^{-1}\mathbf{v} \qquad M^{-1}\ \mathbf{v} = \frac{1}{\lambda}\mathbf{v}$$

5. (a) (i) $28x^3 + 3$ (ii) $-9(1-3x)^2$ (iii) $\dfrac{16}{(1-2x)^3}$ (iv) $\dfrac{3}{2\sqrt{1+3x}}$

(b) (i) $-2e^{-2x}$ (ii) $-2\sin(2x=1)$ (iii) $2e^x(1+e^x)$

(c) (i) $\dfrac{x(5x+4)}{2\sqrt{1+x}}$ (ii) $\dfrac{x^2-4x-1}{(x^2+1)^2}$ (iii) $\text{cosec}x + \sec x$

6.

$y = e^{-x}\cos 4x + e^{-x}\sin 4x$ $y' = -5e^{-x}\cos 4x - 3e^{-x}\sin 4x$ $y'' = -7e^{-x}\cos 4x + 2e^{-x}\sin 4x$

So $y'' - 2y' - 5y = 0$

7. (a) $y' = -6x + 2 \Rightarrow y' = 0, x = \frac{1}{3}, y\left(\frac{4}{3}\right) = -\frac{2}{3},$ $y'' = -6 \Rightarrow$ maximum

(b) $y' = 3x^2 - 18x + 26 \Rightarrow y' = 0, x = 3 \pm \dfrac{\sqrt{3}}{3},$

$y\left(3 + \frac{\sqrt{3}}{3}\right) = -\left(\frac{2}{3\sqrt{3}}\right),$ $y\left(3 - \frac{\sqrt{3}}{3}\right) = \left(\frac{2}{3\sqrt{3}}\right).$

Then $y'' = 6x - 18$ so $y''\left(3 - \frac{\sqrt{3}}{3}\right) < 0$ maximum and

$y''\left(3 + \frac{\sqrt{3}}{3}\right) > 0$ minimum.

Also $y'' = 0$ for $x = 3$, a point of inflection.

8. (a) $y' = -e^{-2x}(1+2x) \Rightarrow y'(0) = -1$ and $y(0) = 1$.

The tangent has gradient -1 and passes through the point $(0,1)$ so

the equation of the tangent is $x + y = 1$

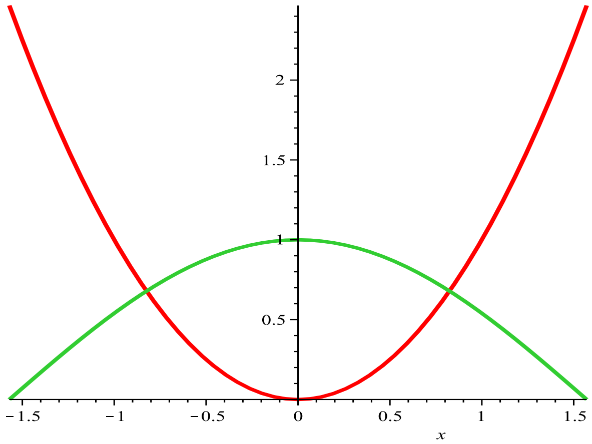
(b) $f(x) = x^3 + x - 1,$ $f'(x) = 3x^2 + 1$ so the iteration formula is

$$x_{n+1} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}$$

We have $f(1) = 1$ and $f(0) = -1$ so the root must lie in $(0,1)$. Take $x_0 = \frac{1}{2}$

Then $x_1 = 0.714286$
$x_2 = 0.683179$
$x_3 = 0.682328$
$x_4 = 0.682327$ so the root is 0.682 to 3 d.p.s.

(c) $f(x) = \cos x - x^2$. From the graph we can see that there are two roots one the negative of the other. Take $x_0 = \frac{\pi}{4}$



$f'(x) = -\sin x - 2x$ so the iteration scheme is

$$x_{n+1} = x_+ \frac{\cos x_n - x^2}{\sin x_n + 2x_n}$$

$x_1 = 0.825021$
$x_2 = 0.824132$
$x_3 = 0.824132$ so the roots are ± 0.8241 to 4 d.p.s.

9. (a)

$$\sum_{n=0}^{1499} 2n + 1 = 1500 + 2\sum_{n=0}^{1499} n = 1500 + \frac{2 \times 1500 \times (0 + 1499)}{2} = 2,250,000$$

(b)

$$\text{(i)} \quad \sum_{n=1}^{10} 3^n = 3\frac{1 - 3^{10}}{1 - 3} = 88572$$

$$\text{(ii)} \quad \sum_{n=0}^{\infty} \frac{2}{3^n} = \frac{2}{1 - \frac{1}{3}} = 3$$

$$\text{(iii)} \quad \sum_{n=4}^{11} 2^n = 2\left(\sum_{n=0}^{11} 2^n - \sum_{n=0}^{3} 2^n = \frac{1 - 2^{12}}{1 - 2} - \frac{1 - 2^4}{1 - 2}\right) = 4080$$

(c)

$$\text{(i)} \quad \binom{7/}{5} \times 3^5 \times (-2)^2 = 20412$$

$$\text{(ii)} \quad \binom{6}{5} \times 4^5 \times (-5) = -30720$$

(d)

$$\text{(i)} \quad \frac{1}{1 + 2x} = 1 - 2x + 4x^2 - 8x^3 \ldots = \sum_{n=0}^{\infty} (-1)^n 2^n x^n \qquad |x| < \frac{1}{2}$$

$$\text{(ii)} \quad \frac{1}{\sqrt{1-3x}} = 1 + \frac{3x}{2} + \frac{27x^2}{8} + \frac{135x^3}{16} \dots \qquad |x| < \frac{1}{3}$$

$$\text{(ii)} \quad e^{-2x} = 1 - 2x + 2x^2 - \frac{4x^3}{3} \dots = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n x^n}{n!} \qquad |x| \in \mathbb{R}$$

$$\text{(iii)} \quad \sin 2x^2 = 2x^2 - \frac{4x^6}{3} + \frac{4x^{10}}{15} + \frac{8x^{14}}{315} \qquad |x| \in \mathbb{R}$$