Solutions to exercises on page 52

1. (a)
$$y = \left(\frac{C - 3e^{-2x}}{2}\right)^{\frac{1}{3}}$$

   (b)
$$y = \frac{1}{x}e^{x-1}$$

   (c)
$$y = \frac{2}{\sqrt{9 - 8\sqrt{1+t^2}}}$$

   (d)
$$y = \tanh\left(\tanh^{-1} 2 - t\right)$$

   (e)
$$y = A\sin^2 x \qquad \text{also} \quad y \equiv 0$$

   (f)
$$y + \ln|1+y| = x^2 + C$$

   (g)
$$e^y(y-1) = A - e^{-x}$$

   (h)
$$\frac{y-2}{y+1} = C\left(\frac{x+1}{x+3}\right)^3$$

   which could also be expressed as

$$y = \frac{(-1-2\,C)\,x^3 + (-9-6\,C)\,x^2 + (-27-6\,C)\,x - 27 - 2\,C}{(-1+C)\,x^3 + (-9+3\,C)\,x^2 + (-27+3\,C)\,x - 27 + C}$$

   (i)
$$y = A\ln\sec(c - x)$$

   (j)
$$y = \frac{c_1x^2}{2} + c_2 \qquad \text{also} \quad y \equiv 0 \quad \text{and} \quad y = C$$

   (k)
$$y = \frac{2}{\sqrt{x}}$$

   (l)
$$y = \frac{2}{(1-x^2)^2}$$

2. (a) Integrating factor is e^x
$$y = \frac{5}{2}e^x + Ce^{-x}$$

(b) Integrating factor is $e^{\int \frac{1}{x} dx} = x$

$$y = \frac{x^4}{5} - \frac{x}{2} + \frac{23}{10x}$$

(c) Integrating factor is $(x+1)^2$.

$$y = \frac{x+1}{3} + \frac{C}{(x+1)^2}$$

(d) Integrating factor is t^2

$$y = \left(\frac{1}{t} - \frac{1}{t^2}\right) e^t + \frac{C}{t^2}$$

(e) Integrating factor is $\frac{1}{x^2}$

$$y = x^2 \ln|x| + Cx^2$$

(f) Integrating factor is e^{x^2}

$$y = -\frac{1}{2} + (x + C)e^{-x^2}$$

(g) Integrating factor is $\sec x$

$$y = \sin x + C \cos x$$

(h) Integrating factor is x^2

$$y = \frac{1}{2} - \frac{1}{x} + \frac{C}{x^2}$$

(i) Integrating factor is $\dfrac{1}{\sqrt{1-x^2}}$

$$y = 1 + C\sqrt{1-x^2}$$

(j) Integrating factor is $\frac{1}{1-x}$

$$y = (1-x)\left(C - \frac{x^2}{2}\right)$$

(k) Integrating factor is xe^x

$$y = e^{-x}\left(1 + \frac{C}{x}\right)$$

(l) Integrating factor is e^x

$$y = \frac{1}{2}e^x + \frac{3}{2}e^{-x}$$

(m) Integrating factor is $\frac{1}{\sqrt{1+x^2}}$, then use the substitution $x = \sinh u$

$$y = x - \frac{1}{\sqrt{2}}\sqrt{1+x^2}$$

(n) Integrating factor is $\frac{1}{x^3}$

$$\text{(a)} \quad y = x^2 - 2x^3, \quad \text{(b)} \quad y = x^2$$

(o) Integrating factor is y^3. $y \equiv 0$ is a solution and

$$x = \frac{4y^2}{5} + \frac{C}{y^3}$$

3. (a) Process to the form $x\dfrac{du}{dx} = \dfrac{u^4}{1-u^3}$

$$\frac{x^3}{y^3} + 3\ln y = 1$$

(b) Process to $u + x\dfrac{du}{dx} = u + \sqrt{1+u^2}$

$$y = \frac{x^2}{2} - 1$$

(c) Process to $u + x\dfrac{du}{dx} = \dfrac{2-u}{1-2u}$

$$y^2 - xy + x^2 = C$$

(d) Process to $u + x\dfrac{du}{dx} = \dfrac{1+u}{1-u}$

$$2\tan^{-1}\frac{y}{x} = \ln(x^2+y^2) + C$$

(e) Put $x = p + A$ and $y = q + B$. Then $A + B = 1$ and $A - B = -5$ so $x = p - 2, y = q + 3$. Then

$$\frac{dp}{dq} = \frac{p-q}{p+q}.$$

Now substitute $q = up$ to get $p\frac{du}{dp} + u = \dfrac{1-u}{1+u}$ which solves to give

$$p^2 - 2qp - q^2 = C \Rightarrow (y-3)^2 + 2(y-3)(x+2) - (x+2)^2 = C$$

(f) In the same way as in the previous example $x = p + \frac{5}{4}, y = q - \frac{7}{4}$ to get

$$\frac{dp}{dq} = \frac{2\frac{p}{q} + 2}{3\frac{p}{q} + 1}$$

Now substitute $q = up$ to get $u + u\dfrac{du}{dp} = \frac{2u+2}{3+u}$. Use partial fractions to get

$$\frac{u+2}{(u-1)^4} = Cp^3 \Rightarrow y + 2x - \frac{3}{4} = C(y - x + 3)^4$$

(g) Let $u = y - t$ then

$$\frac{du}{dt} = (u+1)(u-2)$$

With partial fractions

$$\frac{u+1}{u-2} = Ae^{-3t} \Rightarrow y = \frac{t - 1 + Ae^{-3t}(t+2)}{1 - Ae^{-3t}}$$

(h) Let $y^2 = u$ then

$$\frac{du}{dt} = ut + e^{\frac{t^2}{2}}$$

$$y = e^{\frac{t^2}{4}}\sqrt{t + C}$$

(i) Substitute $\frac{y}{1+t} = u$ to get $\frac{du}{dt} + \frac{u}{t} = t^2$. The integrating factor is t so

$$y = (1+t)\left(\frac{t^3}{4} + \frac{C}{t}\right)$$