Chapter 8 Exercises
Solutions

- **Exercise on p 89**

Outcome	Pairs that give the outcome	Frequency
2	(1,1)	1
3	(1,2),(2,1)	2
4	(1,3),(3,1),(2,2)	3
5	(1,4),(4,1),(2,3),(3,2)	4
6	(1,5),(5,1),(2,4),(4,2),(3,3)	5
7	(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)	6
8	(2,6),(6,2),(3,5),(5,3),(4,4)	5
9	(3,6),(6,3),(4,5),(5,4)	4
10	(4,6),(6,4),(5,5)	3
11	(5,6),(6,5)	2
12	(6,6)	1

There are 36 possible outcomes - i.e. 6^2.

$$E[X] = \frac{1}{36}(2+6+12+20+30+40+42+4+36+30+22+12) = 7$$

The probability of a score of 10 or more is $P(X \geq 10) = \frac{6}{36} = \frac{1}{6}$

- **Exercise on p 94**

$$\begin{pmatrix} 2 & 3 & 1 \\ 1 & 4 & 0 \\ 3 & -2 & -1 \end{pmatrix}$$

For player 2, strategy I is dominated by $\left(\frac{1}{4}\right)$II $+\left(\frac{3}{4}\right)$III. Thus

$$\tfrac{3}{4} + \tfrac{3}{3} = \tfrac{3}{2} < 2$$

$$\tfrac{4}{4} + \tfrac{0}{1} = 1 = 1$$

$$-\tfrac{2}{4} - \tfrac{3}{4} = -\tfrac{5}{4} < 3$$

and we remove strategy III.

$$\begin{pmatrix} 3 & 1 \\ 4 & 0 \\ -2 & -1 \end{pmatrix}$$

Now for player 1, strategy III is dominated by either I or II

$$\begin{pmatrix} 3 & 1 \\ 4 & 0 \end{pmatrix}$$

For player 2 strategy II is dominated by III

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Thus player 1 plays strategy I, player 2 plays strategy III.
The value of the game is 1.

- **Exercise on p 100** Let the strategies for player 1 be (p_1, p_2) and for player 2 be (q_1, q_2).

		Player 2	
		I	II
	I	1	-2
Player 1	II	-2	4

Then

$$p - 2(1-p) = -2p + 4(1-p) \Rightarrow p = \frac{2}{3}.$$

While

$$q - 2(1-q) = -2q + 4(1-q) \Rightarrow q = \frac{2}{3}.$$

Thus the solution is $\left(\left(\frac{2}{3}, \frac{1}{3}\right), \left(\frac{2}{3}, \frac{1}{3}\right)\right)$ and the value of the game is zero.

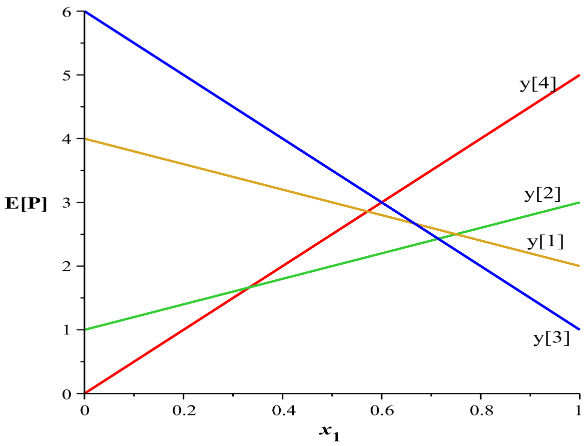
- **First exercise on p 102**

$$A = \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$

Let the strategies for player 1 be (x_1, x_2) and for player 2 be (y_1, y_2, y_3, y_4). Then, for each strategy of player 2 we can calculate the expected payoff for player 1.

	E[P]
y_1	$2x_1 + 4(1-x_1) = 4 - 2x_1$
y_2	$3x_1 + 1(1-x_1) = 1 + 2x_1$
y_3	$x_1 + 6(1-x_1) = 6 - 5x_1$
y_4	$5x_1$

We now plot these expressions for $x_1 \in [0,1]$)



The solution is $\max_{x_1 \in [0,1]}\{1 + 2x_1, 6 - 5x_1\}$, which is $x_1 = \frac{5}{7}$ and the value of the game is $v = \frac{17}{7}$.

Player 2 thus plays only strategies y_2 and y_4, thus - using that $y_2 = 1 - y_4$,

$$(1 + 2x_1)y_4 + (6 - 5x_1)(1 - y_4) \leq \frac{17}{7}.$$

We put $x = 0$

$$y_4 + 6(1 - y_4) \leq \frac{17}{7} \longrightarrow y_4 \geq \frac{5}{7},$$

put $x = 1$

$$3y_4 + (1 - y_4) \leq \frac{17}{7} \longrightarrow y_4 \leq \frac{5}{7},$$

Thus Player 2's strategy is $\left(0, \frac{2}{7}, 0, \frac{5}{7}\right)$.

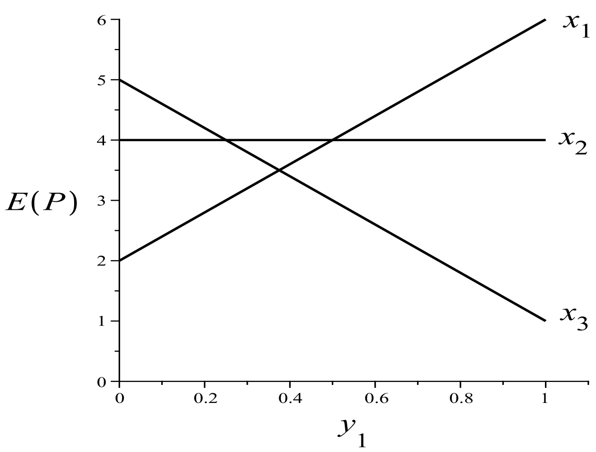
- **Second exercise on p 102**

$$\begin{pmatrix} 1 & 5 \\ 4 & 4 \\ 6 & 2 \end{pmatrix}$$

The strategies for player 1 are (x_1, x_2, x_3) and for player 2 (y_1, y_2).Then

	E[P]
x_1	$5 - 4y_1$
x_2	4
x_3	$4y_1 + 2$

We plot these lines



The solution is found from

$$5 - 4y_1 = 4y_1 - 2 \Rightarrow y_1 = \frac{3}{8}$$

Player 2's strategy is thus $(\frac{3}{8}, \frac{1}{8})$ and the value of the game is $v = \frac{7}{2}$
We find the solution for player 1 by discarding strategy x_2 and solving

$$x_1(5 - 4y_1) + (1 - x_1)(4y_1 + 2) \leq \frac{7}{2}$$

put $y_1 = 0$ we have

$$5x_1 + 2 - 2x_1 \leq \frac{7}{2} \longrightarrow x_1 \leq 1,$$

put $y_1 = 1$ we have

$$x_1 + 6 - 6x_1 \leq \frac{7}{2} \longrightarrow x_1 \geq 1.$$

Hence$x_1 = \frac{1}{2}$ and Player 1's strategy is thus $(\frac{1}{2}, 0, \frac{1}{2})$.

Chapter 8 Exercises

1. Let player 1 have strategies (x_1, x_2) and player 2 (y_1, y_2). Then

$$-x_1 - 2(1 - x_1) = -3x_1 + 2(1 - x_1) \Rightarrow x_1 = \frac{2}{3}$$

The strategy for Player 1 is $\left(\frac{2}{3}, \frac{1}{3}\right)$ and the value of the game is $v = -\frac{4}{3}$
For Player 2 we have

$$-y_1 - 3(1 - y_1) = -2y_1 + 2(1 - y_1) \Rightarrow y_1 = \frac{5}{6}$$

The strategy for Player 2 is $\left(\frac{5}{6}, \frac{1}{6}\right)$.

2. (a)

$$\begin{pmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & 1 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix}$$

For player 2 $\left(\frac{3}{4}\right)$II + $\left(\frac{1}{4}\right)$IV $\leq$ I, thus

$$\tfrac{12}{4} + \tfrac{0}{4} = 3 < 5$$

$$\tfrac{9}{4} + \tfrac{1}{4} = \tfrac{5}{2} < 4$$

$$-\tfrac{3}{4} + \tfrac{3}{4} = 0$$

$$-\tfrac{6}{4} + \tfrac{2}{4} = -1 < 1$$

Removing strategy I we have

$$\begin{pmatrix} 4 & 1 & 0 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \\ -2 & 1 & 2 \end{pmatrix}$$

Now, for Player 1 $\left(\frac{1}{3}\right)$I + $\left(\frac{2}{3}\right)$III $\geq$ III

$$\begin{pmatrix} 4 & 1 & 0 \\ 3 & 2 & 1 \\ -1 & 4 & 3 \end{pmatrix}$$

For Player 2 now III dominates II.

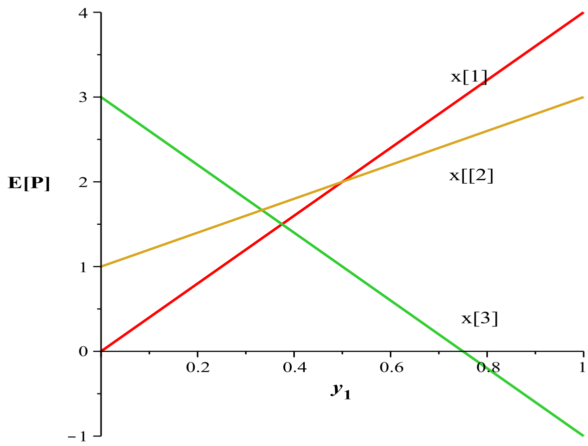
$$\begin{pmatrix} 4 & 0 \\ 3 & 1 \\ -1 & 3 \end{pmatrix}$$

There are no further dominated strategies so we solve the game using the graphical method.

Let player 1 have remaining strategies (x_1, x_2, x_3) and player 2 (y_1, y_2). Then, for player 2

	E[P]
x_1	$4y_1$
x_2	$2y_1 + 1$
x_3	$3 - 4y_1$

We plot the lines and find the solution space:



The solution is where

$$3 - 4y_1 = 4y_1 \Rightarrow y_1 = \frac{3}{8}$$

The strategy for Player 2 is $\left(0, 0, \frac{3}{8}, \frac{5}{8}\right)$ and the value of the game is $v = \frac{3}{2}$

Thus Player 1's strategy is $\left(\frac{4}{5}, 0, \frac{1}{5}, 0\right)$

(b)

$$\begin{pmatrix} 10 & 0 & 7 & 1 \\ 2 & 6 & 4 & 7 \\ 6 & 3 & 3 & 5 \end{pmatrix}$$

For Player 1 we have $\left(\frac{6}{7}\right)\text{II} + \left(\frac{1}{7}\right)\text{III} \leq IV$ so we eliminate strategy IV.

$$\begin{pmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \\ 6 & 3 & 3 \end{pmatrix}$$

Now for Player 1 $\left(\frac{1}{2}\right)$I + $\left(\frac{1}{2}\right)$II $\geq$III so we eliminate strategy III.

$$\begin{pmatrix} 10 & 0 & 7 \\ 2 & 6 & 4 \end{pmatrix}$$

For Player 2 $\left(\frac{1}{2}\right)$I + $\left(\frac{1}{2}\right)$II $\leq$III so we reduce the game to the 2 x 2 matrix

$$\begin{pmatrix} 10 & 0 \\ 2 & 6 \end{pmatrix}$$

Now we have

$$10x_1 + 2(1 - x_1) = 6(1 - x_1) \Rightarrow x_1 = \frac{2}{7}$$

Hence Player 1's strategies are $\left(\frac{2}{7}, \frac{5}{7}, 0\right)$ and the value of the game is $v = \frac{30}{7}$. For Player 2 we get

$$10y_1 = 2y_1 + 6(1 - y_1) \Rightarrow y_1 = \frac{3}{7}$$

Hence Player 2's strategies are $\left(\frac{3}{7}, \frac{4}{7}, 0, 0\right)$

3. The problem for player 1 is

maximise x_4

subject to

$$\begin{aligned}
&x_2 + 9x_3 - x_4 && \geq 0 \\
&7x_1 + 4x_2 - x_3 - x_4 && \geq 0 \\
&2x_1 + 8x_2 - x_3 - x_4 && \geq 0 \\
&4x_1 + 2x_2 + 6x_1 - x_4 && \geq 0 \\
&x_1 + x_2 + x_3 && = 1
\end{aligned}$$

In solver we set the problem up like this

	x_1	x_2	x_3	x_4		constraint
solution	0	0	0	0		
y_1	0	1	9	-1	0	0
y_2	7	4	3	-1	0	0
y_3	2	8	-1	-1	0	0
y_4	4	2	6	-1	0	0
$y_1 + y_2 + y_3 + y_4$					0	1

The solution is $x_1 = 0.08$, $x_2 = 0.51$, $x_3 = 0.42$, while the value of the game is $v = 3.81$. For Player 2 the problem is

maximise y_5

subject to

$$\begin{aligned}
&7y_2 + 2y_3 + 4y_4 - y_5 && \leq 0 \\
&y_1 + 4y_2 + 8y_3 + 2y_4 - y_5 && \leq 0 \\
&9y_1 - 3y_2 - y_3 + 6y_4 - y_5 && \leq 0 \\
&y_1 + y_2 + y_3 + y_4 && = 1
\end{aligned}$$

The solver set up is

	y_1	y_2	y_3	y_4	y_5		constraint
solution	0	0	0	0	0		
x_1	0	7	2	4	0-1	0	0
x_2	1	4	8	2	-1	0	0
x_3	9	3	-1	6	-1	0	0
$x_1 + x_2 + x_3$					0	1	

The solution is $y_1 = 0, \quad y_2 = 0.11, \quad y_3 = 0.26, \quad y_4 = 0.62$ and once more the value of the game is 3.81

Note that we can also solve the game in Maple using the linear programming package with the following commands:

> **with(optimisation)**

> **LPSolve**($x[4], \{x[2] + 9x[3] - x[4] \geq 0, 7x[1] + 4x[2] - x[3] - x[4] \geq 0,$
$2x[1] + 8x[2] - x[3] - x[4] \geq 0, 4x[1] + 2x[2] + 6x[1] - x[4] \geq 0, x[1] + x[2] + x[3] = 1\},$
assume=nonnegative,maximize)

> **LPSolve**($y_5, \{7y_2 + 2y_3 + 4y_4 - y_5 \leq 0, y_1 + 4y_2 + 8y_3 + 2y_4 - y_5 \leq 0$

$9y_1 + 3y_2 - y_3 + 6y_4 - y_5 \leq 0, y_1 + y_2 + y_3 + y_4 = 1$

assume=nonnegative)

4.

$$\begin{pmatrix} 0 & 2 \\ t & 1 \end{pmatrix}$$

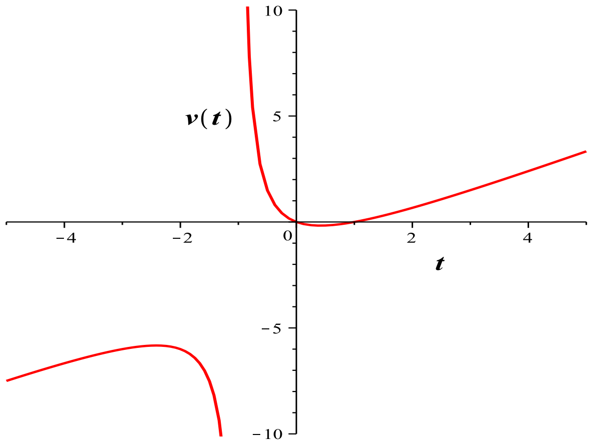
For Player 1 with strategies (x_1, x_2) we have

$$t(1 - x_1) = 2x_1 + (1 - x) \Rightarrow x_1 = \frac{t-1}{t+1}$$

Player I's strategies are $\left(\frac{t-1}{t+1}, \frac{2}{t+1}\right)$. The value of the game is $v(t) = \frac{t(t-1)}{t+1}$.
For Player 2 with strategies (y_1, y_2) we have

$$2(1 - y_1) = ty_1 + (1 - y_1) \Rightarrow y_1 = \frac{1}{1+t}$$

Player 2 has strategies $\left(\frac{1}{1+t}, \frac{t}{1+t}\right)$. Clearly we cannot have $t = 1$.
We plot the graph of t and $v(t)$



It is evident from the graph of $v(t)$ that solutions to the game exist for only a restricted set of values of t. To find this we compute the values of t at which the turning points occur.

$$\frac{dv}{dt} = \frac{t^2 - 1 + 2\,t}{(t+1)^2}$$

Solving $\frac{dv}{dt} = 0$ we find that the domain of v is $\{\mathbb{R} \setminus [-1-\sqrt{2}, -1+\sqrt{2}]\}$.

5.

$$\begin{pmatrix} 0 & 8 & 5 \\ 8 & 4 & 6 \\ 12 & -4 & 3 \end{pmatrix}$$

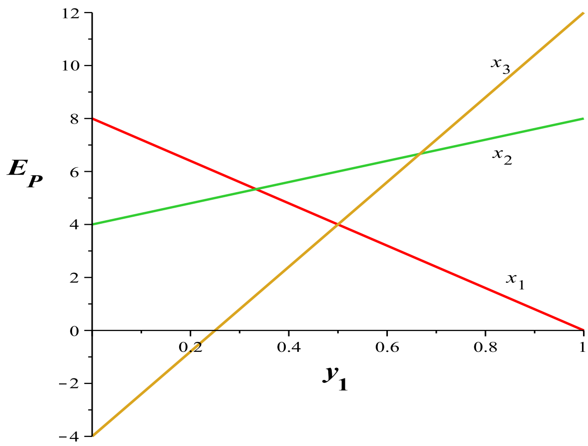
We begin by eliminating strategy III for Player 2 since $\left(\frac{3}{8}\right)\mathrm{I} + \left(\frac{5}{8}\right)\ \mathrm{II} \leq \mathrm{III}$.

A note on how to find the dominating probability combination.
In this case we can see that in strategy I for player 2, while 8 and 12 are the highest values in their rows there is no combination of 8 and 56 in row 1 that can be less than 0, so strategy I cannot be dominated. Similarly, for strategy II for player 2, 8 is the highest value in its row but -4 cannot be greater than any combination of 3 and 13 in the third row. Thus the only candidate for a dominated strategy is III. Then we try weights $w, 1-w$ so that we must have $8(1-w) \leq 5$ in the first row, suggesting that a trial solution for the weights for strategies I and II are $\left(\frac{3}{8}, \frac{5}{8}\right))$
We thus obtain the 3x2 matrix

$$\begin{pmatrix} 0 & 8 \\ 8 & 4 \\ 12 & -4 \end{pmatrix}$$

Thus for Player 2

	E[P]
x_1	$8 - 8y_1$
x_2	$4y_1 + 4$
x_3	$16y - 4$



Plotting the graph it is clear that the solution is to be found at the point where

$$8 - 8y_1 = 16y_1 - 4 \Rightarrow y_1 = \frac{1}{2}$$

Player 2's strategy is thus $\left(\frac{1}{2}, \frac{1}{2}, 0\right)$ and the value of the game $v = 4$
For player 1 we ignore x_2 and solve

$$-4x - 3 + 8(1 - x_3) = 4 \Rightarrow x_3 = \frac{1}{3}.$$

Player 1's strategy is $\left(\frac{2}{3}, 0, \frac{1}{3}\right)$.

6. The set-up in solver is exactly the same process as in the answers to question 3. The solutions are

   (a) Player 1 $\left(\frac{5}{6}, 0, \frac{1}{6}\right)$, Player 2 $\left(\frac{2}{3}, \frac{1}{3}, 0\right)$ and the value of the game is $\frac{1}{3}$.

   (b) This problem appears to defeat Excel Solver. However we can solve it assuming that both player 1 and player 2 play mixed strategies incorporating all of their pure strategies. Let player 1 have strategies (x_1, x_2, x_3) and player 2 (y_1, y_2, y_3). Then, by Theorem 2 (the Equilibrium Theorem) we can write the problem for Player 2 as

$$\begin{aligned} y_2 - 2y_1 3 &= -2y_1 + y + 3 \\ -2y_1 + y_3 &= y_1 - 2y_2 \\ y_1 + y_2 + y_3 &= 1 \end{aligned}$$

   The solution (by Maple or Gaussian elimination) is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ and the value of the game is $-\frac{1}{3}$. For player 1 we have

$$\begin{aligned} x_1 - 2x_2 &= -2x_1 + x_3 \\ x_1 - 2x_2 &= -2x_1 + x_3 \\ x - 1 + x_2 + x_3 &= 1 \end{aligned}$$

   The solution (by Maple or Gaussian elimination) is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$

(c) Player 1 $\left(\frac{1}{4}, 0, \frac{1}{6}\right)$, Player 2 $\left(\frac{2}{5}, \frac{2}{5}, \frac{1}{5}\right)$ and the value of the game is 2.25.

7. The game matrix has a saddle point at entry $a_{32} = 1$ (greater than or equal to all the entries in its column, less than or equal to all the entries in its row. Thus player 1 will play strategy II and player 2 strategy 3, the value of the game is 1 and the matrix method fails because not all pure strategies have a non-zero probability.