MAT 1015
Calculus
Revision Notes

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Introduction

Calculus has been described as the operating system of mathematics. In your first year at Surrey you will study a calculus module which runs for the whole academic year and will underpin much of what you study in your second and third years.

This booklet is intended to provide you with the necessary revision material to ensure that when you start your university mathematics studies what you have learnt for A level is fresh in your minds. You should not need any additional reference material for this revision above what is in these notes. If you have any problems with this material your personal tutor will explain anything you do not fully understand once you start your studies..

When you start at Surrey you will be required to pass a test based on the material in this booklet. This is a computer test which you will be able to take as many times as you need in order to obtain the pass mark of 70%. Students who experience difficulties with the test will get personalised tutorial help.

This booklet

These notes contain numerous examples and exercises. You should aim to do all of the exercises - including the longer questions, before you arrive at Surrey. Solutions to all the exercises are included at the end of this booklet.

1. Algebraic Methods

Algebra is the language of mathematical reasoning. It is important to be fluent in this language and to know its conventions, so that you can read and write mathematics in a way which is correct and easily understood.

Simplification

1. $2a(3b-4)-3b(a-2) \equiv 6ab-8a-3ab+6b \equiv 3ab-8a+6b$
2. $\dfrac{1+x}{2}+\dfrac{2-x}{3} \equiv \dfrac{3(1+x)+2(2-x)}{6} \equiv \dfrac{x+7}{6}$
3. $\dfrac{2}{x}\times\dfrac{3x}{y} \equiv \dfrac{6x}{xy} \equiv \dfrac{6}{y}$
4. $\dfrac{3}{2x}\div\dfrac{6}{x^2} \equiv \dfrac{3}{2x}\times\dfrac{x^2}{6} \equiv \dfrac{x}{4}$
5. $\dfrac{2}{x+3}-\dfrac{1}{x-4} \equiv \dfrac{2(x-4)-(x+3)}{(x+3)(x-4)} \equiv \dfrac{x-11}{(x+3)(x-4)}$
6. $\dfrac{1}{(x+1)(x-1)}-\dfrac{1}{x+1} \equiv \dfrac{1-(x-1)}{(x+1)(x-1)} \equiv \dfrac{2-x}{x^2-1}$

Note that the above have all been written with '$\equiv$' signs, to show that they are *identities*, i.e. they are true for *all* values of x, y, etc. There is nothing wrong with using '$=$' signs, but it is important to know the difference between an identity and an *equation*, which is true only for *certain values* of the symbols.

Examples. Identity: $\sin 2x = 2\sin x\cos x$. Equation: $2x = 4$ (which implies that $x = 2$).

Expansion

1. $(x-2y)(2x-3y) \equiv 2x^2-3xy-4xy+6y^2 \equiv 2x^2-7xy+6y^2$
2. $(x+y)^2 \equiv (x+y)(x+y) \equiv x^2+2xy+y^2$

   You should be able to square a bracket this way without multiplying out. Just add the squares of the terms and twice their product, as in the next examples:
3. $(2p-3q)^2 \equiv (2p)^2+2(2p)(-3q)+(-3q)^2 \equiv 4p^2-12pq+9q^2$
4. $\left(\dfrac{2}{x}+\dfrac{x}{2}\right)^2 \equiv \dfrac{4}{x^2}+2+\dfrac{x^2}{4}$
5. $(x+y+z)^2 \equiv x^2+y^2+z^2+2xy+2yz+2xz$
6. $(w+x+y+z)^2 \equiv w^2+x^2+y^2+z^2+2wx+2wy+2wz+2xy+2yz+2xz$[Note the pattern)]
7. $(x+y)^3 \equiv (x+y)(x^2+2xy+y^2) \equiv x^3+3x^2y+3xy^2+y^3$ [Useful to learn]

8. $(3a - 2b)^3 \equiv (3a)^3 + 3(3a)^2(-2b) + 3(3a)(-2b)^2 + (-2b)^3$ [using the last result]
   $\equiv 27a^3 - 54a^2b + 36ab^2 - 8b^3$

9. $\left(x + \frac{1}{x}\right)^3 = \left(x^3 + \frac{1}{x^3}\right) + 3\left(x + \frac{1}{x}\right)$

Factorisation

To *factorise* an expression means to write it as the *product* of its factors. Look for any *common factors* first, then see if one of the standard methods can be used: grouping in pairs; trinomials (quadratic-type expressions); difference of two squares; and the sum and difference of two cubes. Always check that your answer cannot be factorised any further.

1. $12x^2y + 18xy^3 \equiv 6xy(2x + 3y^2)$
2. $4ax - 2ay - 6bx + 3by \equiv 2a(2x - y) - 3b(2x - y) \equiv (2a - 3b)(2x - y)$
3. $x^2 - 11x + 10 \equiv (x - 1)(x - 10)$
4. $2x^2 + 7xy - 15y^2 \equiv (2x - 3y)(x + 5y)$
5. $x^2 + 2xy + y^2 \equiv (x + y)(x + y) \equiv (x + y)^2$
6. $x^2 - y^2 \equiv (x + y)(x - y)$ [Difference of Squares]
7. $32m^2 - 72n^2 \equiv 8(4m^2 - 9n^2) \equiv 8(2m + 3n)(2m - 3n)$, using the previous result
8. $x^3 + y^3 \equiv (x + y)(x^2 - xy + y^2)$ [Sum of two cubes]
9. $x^3 - y^3 \equiv (x - y)(x^2 + xy + y^2)$ [Difference of two cubes]
10. $8p^3 - 27q^3 \equiv (2p - 3q)(4p^2 + 6pq + 9q^2)$, using the previous result
11. $x^4 - y^4 \equiv (x^2 + y^2)(x^2 - y^2) \equiv (x^2 + y^2)(x + y)(x - y)$
12. $x^4 - 8x^2 - 9 \equiv (x^2 - 9)(x^2 + 1) \equiv (x - 3)(x + 3)(x^2 + 1)$
13. $x^3 - 3x^2 + 4x - 2$: notice that this is 0 when $x = 1$, so $x - 1$ must be a factor. Divide by $x - 1$ to get $(x - 1)(x^2 - 2x + 2)$, which *in this case* does not factorise further.
14. $(x - 1)(x^2 + 2) + (2x - 2)(x^2 + 1) \equiv (x - 1)[(x^2 + 2) + 2(x^2 + 1)] \equiv (x - 1)(3x^2 + 4)$
    [Spot the common factor $(x - 1)$ at the start and so avoid multiplying out.]

Completing the Square

This method is useful for solving quadratic equations (it's how we get the quadratic formula) and also for finding the maximum or minimum of a quadratic without differentiating.

1. $x^2 - 8x + 12 \equiv x^2 - 8x + 16 - 4 \equiv (x - 4)^2 - 4$ [add and subtract 4]
   Hence $x^2 - 8x + 12 = 0$ when $(x - 4)^2 = 4$, i.e. $x - 4 = \pm 2$ so $x = 2$ or $x = 6$
   $x^2 - 8x + 12$ has minimum value -4, when $(x - 4)^2 = 0$, i.e. when $x = 4$
2. $5 - 4x - x^2 \equiv -(x^2 + 4x - 5) \equiv -((x + 2)^2 - 9) \equiv 9 - (x + 2)^2$,
   so $5 - 4x - x^2$ has maximum value 9, when $x = -2$
3. $2x^2 - 8x + 13 \equiv 2(x^2 - 4x) + 13 \equiv 2((x - 2)^2 - 4) + 13 \equiv 2(x - 2)^2 + 5$
4. $x^4 + 2x^2 + 11 \equiv (x^2 + 1)^2 + 10$

Algebraic Long Division; Factor and Remainder Theorems

1. $\dfrac{x^2+4}{x+1} \equiv\equiv \dfrac{x^2-1+5}{x+1} \equiv \dfrac{(x+1)(x-1)+5}{x+1} \equiv x-1+\dfrac{5}{x+1}$
2. $(6x^3-3x^2+5x+4) \div (2x+1) \equiv 3x^2-3x+4$, by Algebraic Long Division. See a textbook or ask your tutor for help if you can't remember how this is done.
3. $(x^3-2x^2+3x+4) \div (x+2) \equiv x^2-4x+11-\dfrac{18}{x+2}$
4. $x+3$ is a factor of $\mathrm{f}(x) \equiv x^4+2x^3-27$ by the Factor Theorem, since $\mathrm{f}(-3)=0$
5. The remainder when $\mathrm{g}(x) \equiv x^3+x^2+1$ is divided by $x+2$ is equal to $\mathrm{g}(-2)=-3$

The method in the last example is the **Remainder Theorem**: to find the numerical remainder when a polynomial is divided by $(ax+b)$, substitute $x=-b/a$ in the expression. It's much quicker than doing the division, if the remainder is all you need to find.

Partial Fractions

A numerical fraction can be split up into a sum of two fractions in infinitely many ways (e.g. $\frac{1}{2}=\frac{1}{4}+\frac{1}{4}=\frac{1}{8}+\frac{3}{8}=...$), but with algebraic fractions this can be done in just one way.

At this stage there will only be *linear* factors in the denominator. For each factor $(ax+b)$ in the denominator, there is a partial fraction $\dfrac{p}{ax+b}$.

If the degree of the numerator is *not less* than that of the denominator, *divide* first.

1. To express $\dfrac{x}{x^2-x-12}$ in partial fractions, let it be $\dfrac{p}{x-4}+\dfrac{q}{x+3}$.

   Then $p(x+3)+q(x-4) \equiv x$, so $p=\frac{4}{7}, q=\frac{3}{7}$ by comparing the coefficient of $x, p+q=1$ and the constant term. $3p-4q=0$.

   Answer: $\dfrac{x}{x^2-x-12} \equiv \dfrac{4}{7(x-4)}+\dfrac{3}{7(x+3)}$.

2. $\dfrac{x^2+3}{(x-1)(x-2)(x+3)} \equiv \dfrac{p}{x-1}+\dfrac{q}{x-2}+\dfrac{r}{x+3}$

   $\equiv \dfrac{p(x-2)(x+3)+q(x-1)(x+3)+r(x-1)(x-2)}{(x-1)(x-2)(x+3)}$

   so $p(x-2)(x+3)+q(x-1)(x+3)+r(x-1)(x-2) \equiv x^2+3$.

   Put $x=1$: $-4p=4$. Put $x=2$: $5q=7$. Put $x=-3$: $20r=12$.

   Hence $p=-1, q=7/5, r=3/5$, giving $\dfrac{-1}{x-1}+\dfrac{7}{5(x-2)}+\dfrac{3}{5(x+3)}$.

3. $\dfrac{x^3+3}{x^2-1} \equiv x+\dfrac{x+3}{x^2-1} \equiv x+\dfrac{p}{x+1}+\dfrac{q}{x-1}$, so $p(x-1)+q(x+1) \equiv x+3$.

   Then $p=-1, q=2$, so $\dfrac{x^3+3}{x^2-1} \equiv x-\dfrac{1}{x+1}+\dfrac{2}{x-1}$.

Linear Equations

1. $3(x+4)-2(3-2x)=4x-9 \quad\Rightarrow\quad 7x+6=4x-9 \quad\Rightarrow\quad 3x=-15 \quad\Rightarrow\quad x=-5$
2. $\dfrac{x+2}{3}=\dfrac{2x-1}{5} \quad\Rightarrow\quad 5(x+2)=3(2x-1) \quad\Rightarrow\quad x=13$
3. $5x+3y=7, 3x-y=1$. Multiply the second equation by 3 to get $9x-3y=3$. Add to the first equation : $14x=10$ so $x=\dfrac{5}{7}$. Substitute back to get $y=\dfrac{8}{7}$.

   The solutions of these *simultaneous* linear equations give the x and y coordinates of the point where two straight lines intersect.

Quadratic and higher-order Equations

The values of x which make an equation true are called the *solutions* or *roots* of the equation. The values of a function $\mathrm{f}(x)$ for which $\mathrm{f}(x)=0$ can also be called the *zeros* of f.

1. $x^2-18x+45=0$. By factorisation, $(x-3)(x-15)=0$. The roots are $x=3, x=15$.
2. $x^2-18x+70=0$. By Completing the Square: $(x-9)^2-11=0$, so $x=9\pm\sqrt{11}$.
3. $3x^2-16x+11=0$. By the formula, $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}=\dfrac{16\pm\sqrt{124}}{6}=\dfrac{8\pm\sqrt{31}}{3}$.
4. $(x+2)(x^2+4x+5)=2(x+2) \Rightarrow (x+2)(x^2+4x+5-2)=0 \Rightarrow (x+2)(x+1)(x+3)=0 \Rightarrow x=-1, x=-2, x=-3$. [Do not divide each side of the original equation by $(x+2)$, as you would not get the solution $x=-2$.]
5. *Non-linear* simultaneous equations: make one unknown the subject of the simpler equation and substitute in the other.

   e.g. $xy=3, x^2+y=8$. Write $y=8-x^2$, so $x(8-x^2)=3$, i.e. $x^3-8x+3=0$. Notice a root $x=-3$; divide by $x+3$ to get $(x+3)(x^2-3x+1)=0$.

   Hence $x=-3$ or $x=\dfrac{3\pm\sqrt{5}}{2}$. Solutions are approx. $(-3,-1), (0.38, 7.85), (2.62, 1.15)$. These are the points of intersection of the two graphs.

The formula for solving $ax^2+bx+c=0$ shows that this quadratic equation has two real roots if $b^2-4ac>0$, one (repeated) real root if $b^2-4ac=0$ and no real roots if $b^2-4ac<0$.

Example Find a condition for $9x^2-8kx+4=0$ to have two distinct real roots.

We need $64k^2>4(9)(4)$, so $4k^2>9$, so $k<-\frac{3}{2}$ or $k>\frac{3}{2}$.

If $ax^2+bx+c=0$, $a\neq 0$, has roots $x=\alpha, x=\beta$ then the equation must be $(x-\alpha)(x-\beta)=0$, i.e. $x^2-(\alpha+\beta)x+\alpha\beta=0$. Comparing this with the original equation, which is $x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$, we see that $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$. These formulae are very easy to remember and very useful.

Example Find the sum and the product of the roots of $5x^2-90x-2000=0$.

The sum is $-\dfrac{b}{a}=-\dfrac{-90}{5}=18$ and the product is $\dfrac{c}{a}=\dfrac{-2000}{5}=-400$.

Inequalities

If an inequality is multiplied or divided through by a *negative* quantity, then the inequality sign must be *reversed.*
Hence $2 < 3$ but $-2 > -3$.It is therefore *not* safe to multiply both sides by an algebraic quantity unless it is known to be positive.
If the reciprocals of both sides are taken the the inequality sign is also *reversed.*
$2 < 3$ but $\frac{1}{2} > \frac{1}{3}$
Otherwise inequalities may be treated like equations.

1. $2x - 7 < 8x + 5 \Rightarrow -6x < 12 \Rightarrow x > -2$
2. $|x - 3| > 5 \Rightarrow x - 3 > 5$ or $x - 3 < -5 \Rightarrow x < -2$ or $x > 8$
3. $x^2 - 8x + 12 \leq 0 \Rightarrow (x - 2)(x - 6) \leq 0 \Rightarrow 2 \leq x \leq 6$. A graph will help. Sketch $y = x^2 - 8x + 12$ and read off the values of x where it lies *below or on* the x-axis.
4. $x^2 - 8x + 12 > 0 \Rightarrow (x - 2)(x - 6) > 0 \Rightarrow x < 2$ or $x > 6$ (where the graph is *above* the x-axis). Do not write nonsense like $2 > x > 6$, since 2 is not > 6 !!
5. $\dfrac{x}{2} \geq \dfrac{2}{x} \Rightarrow x > 0$ and $x^2 \geq 4$, or $x < 0$ and $x^2 \leq 4$. Solution is $x \geq 2$ or $-2 \leq x < 0$.

   Alternatively, sketch the graphs of $y = \dfrac{x}{2}$ and $y = \dfrac{2}{x}$ and find the set of values of x for which the first graph lies *on or above* the second graph.

Transforming Formulae

1. If $2ay + 3bx = c$ then $3bx = c - 2ay$ so $x = \dfrac{c - 2ay}{3b}$
2. If $\dfrac{x^2}{a} = p - q$ then $x^2 = a(p - q)$ so $x = \pm\sqrt{a(p - q)}$.
3. If $2ax + 3by = cx + dy$ then $2ax - cx = dy - 3by$, so $x(2a - c) = y(d - 3b)$, giving $x = \dfrac{y(d - 3b)}{2a - c}$
4. If $(x - a)(x - b) = ab$ then $x^2 - (a + b)x = 0$ so $x(x - [a + b]) = 0$ so $x = 0$ or $x = a + b$

Exercises for Section 1

1. Write each of the following expressions in its simplest form without brackets:

   (a) $5\{x - 4[y + 3(z - 6)]\}$ (b) $\frac{1}{4}[(x+y)^2 - (x-y)^2]$

   (c) $(x - y)(x^2 + xy + y^2)$ (d) $x(y - 2z) - y(z - 2x) - z(x - 2y)$

2. Express as single algebraic fractions in their simplest forms :

   (a) $\dfrac{x^7y^5}{y^8x^3}$ (b) $\dfrac{6x}{12x^2 + 18xy}$

   (c) $\dfrac{8y^2}{y+2} \div \dfrac{2y^3}{y^2 + 3y + 2}$ (d) $\dfrac{x+2}{6} + \dfrac{3x-4}{9}$

   (e) $\dfrac{x+1}{x} + \dfrac{x}{x+1}$ (f) $\dfrac{2x}{x^2-1} - \dfrac{3}{x+1}$

3. Complete the square for each of the following expressions. Hence find the maximum or minimum value of each, and the value of x at which this occurs.

   (a) $x^2 - 8x - 7$, (b) $11 + 2x - 4x^2$, (c) $x^4 + 4x^2 + 7$.

4. Factorise the following as far as possible:

   (a) $x^2 + 5x + 6$ (b) $x^4 - x^2 - 12$ (c) $4ax - 6ay - 12x^2 + 18xy$

   (d) $4x^2 + 8x + 3$ (e) $x^3 - 2x^2 - 5x + 6$ (f) $121p^2 - 169q^2$

   (g) $x^3 + 64y^3$ (h) $x^4 - 16x^2$ (i) $(4x - 3y)^2 - (2x + 3y)^2$

   (j) $(x^2 - 4)(x + 3) - (x + 2)(x^2 + x - 6)$ (Do not multiply the brackets out!)

5. Perform the following divisions:

   (a) $(x^2 - 9x + 20) \div (x - 4)$ (b) $(x^2 + x + 1) \div (x + 1)$

   (c) $(2x^3 - 7x^2 + 8x - 8) \div (2x - 3)$ (d) $(x^3 + 2x^2 + x + 5) \div (x^2 + x + 1)$

6. Express the following in partial fractions:

   (a) $\dfrac{5x+1}{x^2+x-2}$ (b) $\dfrac{x(x+31)}{(x+1)(x-4)(2x-1)}$ (c) $\dfrac{x^3-1}{(x-2)(x-3)}$

7. Answer the following **without** doing any algebraic division.

   (a) Show that $2x - 1$ is a factor of $2x^4 - x^3 - 8x + 4$.

   (b) Find the remainder when $x^3 - 7x^2 + 9$ is divided by $x + 1$.

   (c) $f(x) = x^3 + ax^2 + bx - 1$. When $f(x)$ is divided by $x + 2$, the remainder is 3. When $f(x)$ is divided by $x - 3$, the remainder is 8. Find the values of a and b.

8. Solve the following equations by the method indicated.

   (a) $\dfrac{x-2}{3} + \dfrac{1-2x}{5} = \dfrac{7}{10}$, by multiplying through by 30.

   (b) $2x+5y = -9, -4x-3y = 4$ by any method you know for simultaneous equations.

   (c) $x^2 - 8x - 13 = 0$, by completing the square.

   (d) $x^3 - 7x + 6 = 0$, by dividing by an obvious factor.

   (e) $xy = 7, x + y^2 = 50$, by forming a cubic equation in y.

9. (a) Find the range of values of k for which $kx^2 - 2kx + 1 = 0$ has no real roots.

   (b) Without solving the equation, find in terms of p the sum and the product of the roots of the equation $2px^2 - 4p^2x = 6p^3$, where $p \neq 0$.

10. Solve the following inequalities for x

    (a) $5x + 2 \geq 2x + 1$ (b) $(4x - 2)(3 - x) < 0$ (c) $x^2 + 2x \geq 15$

    (d) $\dfrac{1}{x+1} < 2$ (e) $|x - 3| < 4$

11. In each of the following, express x in terms of the other letters.

    (a) $ax + b = cx + d$ (b) $\sqrt{p(x+p)} = x - p$ (c) $x = kx(1 - x)$

    (d) $x^2 - 4xy + 4y^2 = z^2$ [Hint : factorise the left-hand side.]

12. Rewrite the following expressions using the equations given and simplify fully:

    (a) $x^2 + y^2$ where $x = \dfrac{at}{\sqrt{1+t^2}}, y = \dfrac{a}{\sqrt{1+t^2}}$

    (b) $\dfrac{s^2 - 1}{s^2 - t^2}$ where $s = \sqrt{1 - x^2}, t = \sqrt{1 + x^2}$

    (c) $\dfrac{(p^2 + q^2)}{(p^2 - q^2)}$ where $p = \sqrt{q^2 - 1}$ (d) $x^2 + \dfrac{36}{x^2}$ where $x + \dfrac{6}{x} = 5$

2. Series, Functions, etc.

Sequences and Series

A **sequence** or **progression** is an ordered list of numbers. A **series** is formed by adding the successive terms of the sequence. For example : 1, 2, 4, 8, ... is a sequence, but 1 + 2 + 4 + 8 + ... is a series.

Sigma notation: $\sum_{k=a}^{b} \mathrm{f}(k)$ means the sum of all terms of the form f(k) as k takes integer values from a to b inclusive.

- The Arithmetic Progression (A.P.) with first term a and common difference d is $a, a+d, a+2d, ...$ The nth term is $T_n = a + (n-1)d$.

  Sum of first n terms is $S_n = \frac{n}{2}(2a + (n-1)d)$.

  Example : $\sum_{k=0}^{12}(4k-5) = (-5) + (-1) + ... + 43 = \frac{13}{2}(-10 + 12(4)) = 247$

- The Geometric Progression (G.P.) with first term a and common ratio r is $a, ar, ar^2, ...$ The nth term is $T_n = ar^{n-1}$.

  Sum of first n terms is $S_n = \frac{a(1-r^n)}{1-r}$.

  If $|r| < 1$, the series converges to the *sum to infinity* $S_\infty = \frac{a}{1-r}$

  Example : $\sum_{t=0}^{\infty} 3(2^{-t}) = 3 + \frac{3}{2} + \frac{3}{4} + ... = \frac{3}{1-1/2} = 6$

The Binomial Theorem

The expansions of $(x+y)^2$ and $(x+y)^3$ can be generalised to give the **Binomial Theorem**. Let n be a positive integer. Then

$$(x+y)^n \equiv x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \cdots + y^n.$$

The coefficient of $x^{n-r}y^r$ is $\binom{n}{r} \equiv \frac{n!}{r!(n-r)!}$. Note that $\binom{n}{r} \equiv \binom{n}{n-r}$ and $0! = 1$.

The *binomial coefficients* $\binom{n}{r}$ can also be worked out from **Pascal's Triangle**,

whose rows are 1, 1 1, 1 2 1, 1 3 3 1, 1 4 6 4 1, 1 5 10 10 5 1, etc.

1. $(2x - 3y)^5$
   $\equiv (2x)^5 + 5(2x)^4(-3y) + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3 + 5(2x)(-3y)^4 + (-3y)^5$
   $\equiv 32x^5 - 240x^4y + 720x^3y^2 - 1080x^2y^3 + 810xy^4 - 243y^5$

2. The term in a^5b^4 in the expansion of $\left(3a - \frac{1}{2}b\right)^9$ is

   $$\binom{9}{4}(3a)^5\left(\frac{-b}{2}\right)^4 = \frac{9!}{4!5!}\frac{243a^5b^4}{16} = \frac{126 \times 243a^5b^4}{16} = \frac{15309a^5b^4}{8}$$

Binomial Series

If n is a non-negative integer, the binomial expansion terminates; otherwise, it does not. In the case where it does not terminate, it takes the form of the infinite series

$$(1+x)^n \equiv 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

which is convergent provided that $|x| < 1$. The first term in the bracket *must* be 1.

When the second term in the bracket is small enough, the first few terms of the series give a good approximation to the function being expanded.

1. $(1-2x)^{-2} \equiv 1 - 2(-2x) + \dfrac{(-2)(-3)}{2}(-2x)^2 + \dfrac{(-2)(-3)(-4)}{6}(-2x)^3 + ...$
   $\equiv 1 + 4x + 12x^2 + 32x^3 + ...$ Valid for $|-2x| < 1$, i.e. $-\frac{1}{2} < x < \frac{1}{2}$.
2. $(4+12x)^{1/2} \equiv 4^{1/2}(1+3x)^{1/2} = 2\left(1 + \dfrac{3}{2}x - \dfrac{9}{8}x^2 + ...\right) \equiv 2 + 3x - \dfrac{9}{4}x^2 + ...$
   Valid for $|3x| < 1$, i.e. $-\frac{1}{3} < x < \frac{1}{3}$.
3. $\dfrac{1}{1-x} \equiv (1-x)^{-1} \equiv 1 + x + x^2 + x^3 + ...$
4. $\dfrac{1}{1+x} \equiv (1+x)^{-1} \equiv 1 - x + x^2 - x^3 + ...$
   It's worth knowing these last two, which are valid for $|x| < 1$. You can easily check them by multiplying out: $(1-x)(1+x+x^2+...) \equiv 1$. (Why?)

Indices and Surds

For $x \neq 0$ we define $x^0 \equiv 1,\ x^{-1} \equiv \dfrac{1}{x},\ x^{-2} \equiv \dfrac{1}{x^2}$, etc.

Roots are given by fractional powers, so $x^{1/2} \equiv \sqrt{x}, x^{1/3} \equiv \sqrt[3]{x}, 2x^{-1/2} \equiv \dfrac{2}{\sqrt{x}}, x^{3/4} \equiv (\sqrt[4]{x})^3$.

Note that $\sqrt{x}$ or $x^{\frac{1}{2}}$ means the **positive** real square root of x. Thus $\sqrt{x^2} = x$ if $x \geq 0$, but $-x$ if $x < 0$. Never write things like $\sqrt{4} = \pm 2$.

A number written as an unevaluated root, like $\sqrt{2}$, is said to be in *surd form.* This is exact, whereas any decimal approximation is not.

Laws of indices: $x^a \times x^b \equiv x^{a+b},\ x^a \div x^b \equiv x^{a-b}$.

If $x > 0$ then $(x^a)^b \equiv x^{ab}$. Note that this may fail if $x < 0$, e.g. $[(-2)^2]^{1/2} = 4^{1/2} = 2 \neq (-2)^1$ so $(x^a)^b \neq x^{ab}$ here!

1. $27^{2/3} = 3^2 = 9,$ $\quad 16^{-3/4} = \dfrac{1}{(\sqrt[4]{16})^3} = \dfrac{1}{8}$
2. $x^5 \times x^{-2} \equiv x^3,$ $\quad x^2 \div x^{-3} \equiv x^{2-(-3)} \equiv x^5$
3. $(x^4)^{-2} \equiv x^{-8}$, or $\dfrac{1}{x^8}\quad (x \neq 0)$ $\quad (x^2)^{\frac{3}{4}} \equiv x^{\frac{3}{2}}$, or $\sqrt{x^3}\quad (x > 0)$
4. $\dfrac{x+1}{\sqrt{x^2-1}} \equiv \dfrac{x+1}{\sqrt{x+1}\sqrt{x-1}} \equiv \dfrac{\sqrt{x+1}}{\sqrt{x-1}}$, provided $x > 1$. [What is it if $x < -1$?]

5. $\sqrt{4x^2+4x+1} \equiv 2x+1$ if $x \geq -\dfrac{1}{2}$, but $-(2x+1)$ if $x < -\dfrac{1}{2}$.

6. $\dfrac{2}{\sqrt{3}} = \dfrac{2\sqrt{3}}{\sqrt{3}\sqrt{3}} = \dfrac{2\sqrt{3}}{3}$ (Rationalising the denominator.)

7. $\dfrac{2+3\sqrt{2}}{3-4\sqrt{2}} = \dfrac{(2+3\sqrt{2})(3+4\sqrt{2})}{(3-4\sqrt{2})(3+4\sqrt{2})} = \dfrac{-30-17\sqrt{2}}{23}$

Exponential and Logarithmic Functions

An expression of the form a^x is called an **exponential** function of x. A special case, known as *the* exponential function, is $\exp(x)$ or e^x, where $\exp(1)$ or e is about 2.718.

If $x = a^y$, there is no simple way of making y the subject of this formula so for $a > 0$ we define the **logarithm** of x, to the **base** a, by $\log_a x = y$ where $a^y = x$.

In other words, the logarithm of a number is the power to which the base must be raised to give that number. In the case $a = e$, we write $\ln x$ for $\log_e x$ and call this the **natural logarithm** of x. Note that $\ln(e^x) = e^{\ln x} = x$.

Negative numbers and zero do not have real logarithms. Because logarithms are powers of a positive number, the laws of indices apply. Hence we have these **laws of logarithms**:

- $\log_a xy = \log_a x + \log_a y$, (Note that $\log_a(x+y)$ cannot be expanded.)
- $\log_a \dfrac{x}{y} = \log_a x - \log_a y$,
- $\log_a x^n = n\log_a x$,
- $\log_b x = \log_a x \div \log_a b$ (Change of Base rule).

Equations in which the unknown appears as a power can often be solved by taking logarithms, to a suitable base, of both sides.

1. $\log_3 9 = 2$; $\log_a 1 = 0$; $\log_4 2 = \dfrac{1}{2}$; $\log_2 \dfrac{1}{8} = -3$
2. $\ln e^4 = 4$; $\exp(\ln 3) = 3$; $\log_3 5 = \ln 5 \div \ln 3 \approx 1.465$
3. If $\log_3 x = 4$ then $x = 3^4 = 81$. If $\ln(x+1) = 2$ then $x = e^2 - 1$.
4. If $2\log_3 x + \log_3(x-2) = 2$ then $\log_3[x^2(x-2)] = 2$ so $x^3 - 2x^2 = 9$ Hence $x = 3$.
5. If $3^{x+1} = 2^{x-2}$ then, taking natural logs of both sides, $(x+1)\ln 3 = (x-2)\ln 2$, so $(\ln 2 - \ln 3)x = \ln 3 + 2\ln 2$, so $x \approx -6.13$.
6. If $5^{2x} - 5^{x+1} + 4 = 0$, we have a quadratic in 5^x. It is $(5^x)^2 - 5(5^x) + 4 = 0$, so $(5^x - 1)(5^x - 4) = 0$. Thus $5^x = 1$ or $5^x = 4$, so $x = 0$ or $x = \log_5 4 \approx 0.861$.

Trigonometric Functions

You should know the definitions of sine, cosine and tangent as ratios of sides in a right-angled triangle. These can be extended using their periodic properties, and illustrated using the four quadrants of a unit circle. Hence the trigonometric functions are sometimes called the **circular functions**. In the context of calculus, all angles are assumed to be measured in *radians*. There are 2π radians in a complete rotation, so 1 radian $= \frac{180}{\pi}^\circ$ and $1^\circ = \frac{\pi}{180}$ radians. We often express angles in terms of π.

You should learn, or be able to work out quickly, conversions such as:

Degrees	0	30	45	60	90	120	135	150	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

and also the following exact trigonometric ratios:

	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Really we should not write $\tan\frac{\pi}{2} = \infty$, but $\tan x \to \infty$ as $x \to \frac{\pi}{2}$.

Note that $\tan x \equiv \dfrac{\sin x}{\cos x}$.

The secant, cosecant and cotangent functions are defined by :

$$\sec x \equiv \frac{1}{\cos x},\ \operatorname{cosec} x \equiv \frac{1}{\sin x},\ \cot x \equiv \frac{1}{\tan x}.$$

It follows from $\sin^2 x + \cos^2 x \equiv 1$ (the trigonometric form of Pythagoras' Theorem) that

$$\tan^2 x + 1 \equiv \sec^2 x \quad \text{and} \quad 1 + \cot^2 x \equiv \operatorname{cosec}^2 x.$$

You should know the following trigonometric identities:

$$\sin(A \pm B) \equiv \sin A \cos B \pm \sin B \cos A$$

($\mp$ means $-$ when $\pm$ is $+$, and vice-versa)

$$\cos(A \pm B) \equiv \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm B}{1 - \mp \tan A \tan B}$$

From them we can deduce these *factor formulae*:

$$\sin x \pm \sin y \equiv 2\sin\left(\frac{x \pm y}{2}\right)\cos\left(\frac{x \mp y}{2}\right), \quad (\mp \text{ means } - \text{ when } \pm \text{ is } +, \text{ and vice-versa})$$

$$\cos x + \cos y \equiv 2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right), \quad \cos x - \cos y \equiv -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right),$$

and *product formulae*:

$$\sin A \cos B = \frac{1}{2}\left(\sin(A+B) + \sin(A-B)\right)$$

$$\cos A \cos B \equiv \frac{1}{2}\left(\cos(A+B) + \cos(A-B)\right)$$

$$\sin A \sin B \equiv \frac{1}{2}\left(\cos(A-B) - \cos(A-B)\right).$$

We also have the following *double angle formulae*

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1-\tan^2 A}.$$

and the corresponding *half angle formulae*

$$\sin\frac{A}{2} \equiv \sqrt{\frac{1}{2}(1-\cos A)}, \quad \cos\frac{A}{2} \equiv \sqrt{\frac{1}{2}(1+\cos A)}, \quad \tan\frac{A}{2} \equiv \sqrt{\frac{1-\cos A}{1-\cos A}}.$$

We can also express

$$\alpha\sin A + \beta\cos A \equiv \sqrt{\alpha^2+\beta^2}\sin\left(A + \arctan\frac{\beta}{\alpha}\right)$$

$\arctan x$ is read as "the angle whose tangent is x".

1. Prove that $\sin x\cos y \equiv \frac{1}{2}(\sin(x+y)+\sin(x-y))$.
   $\sin(x+y) \equiv \sin x\cos y + \cos x\sin y$ and $\sin(x-y) \equiv \sin x\cos y - \cos x\sin y$.
   Adding these and dividing by 2 gives the required result.
2. If $\sin x = \frac{3}{5}$ and $\sin y = \frac{5}{13}$, where $0 < x < \frac{\pi}{2}, 0 < y < \frac{\pi}{2}$, find $\tan(x+y)$.

   Using the $3, 4, 5$ and $5, 12, 13$ triangles we see that $\tan x = \frac{3}{4}, \tan y = \frac{5}{12}$, so

   $\tan(x+y) = \dfrac{3/4 + 5/12}{1 - 3/4 \times 5/12} = \dfrac{56}{33}$
3. Find $\cos\frac{\pi}{12}$ in surd form

   $\cos\frac{\pi}{12} = \cos\left(\frac{\pi}{4}-\frac{\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{3} + \sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{1}{\sqrt{2}}\left(\frac{1}{2}+\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}+1}{2\sqrt{2}}.$

Trigonometric Equations

Using the periodic properties of the trig. functions, we can find all solutions of a trig. equation in a given interval. If no interval is specified then generally there are infinitely many solutions.

A formula for these solutions in terms of a variable integer n is called the **general solution**.

1. Solve the equation $\sec x = 2$ (a) for $0 \le x < 2\pi$, (b) generally.

   (a) If $\sec x = 2$ then $\cos x = \frac{1}{2}$.

   Cosines are positive in the first and fourth quadrants, so $x = \frac{\pi}{3}$ or $x = \frac{5\pi}{3}$.

   (b) The generalisation of this is that x is either of the above angles plus or minus an integer multiple of 2π. More simply, $x = 2n\pi \pm \frac{\pi}{3}, n \in \mathbb{Z}$. ($\mathbb{Z}$ = set of integers.)

2. Find the general solution of the equation $\cos^2 x - \sin x - 1 = 0$.

   We can re-write this as $1 - \sin^2 x - \sin x - 1 = 0$, so $\sin x(1 + \sin x) = 0$, giving $\sin x = 0$ or $\sin x = -1$. Hence $x = n\pi$ or $x = 2n\pi - \frac{\pi}{2}, n \in \mathbb{Z}$.

3. Solve the equation $\cos 3x + \cos x = \frac{1}{2}$.

   By the factor formulae (see above), we get $2\cos 2x \cos x = \frac{1}{2}$, so $4\cos x(2\cos^2 x - 1) = 1$, so $8\cos^3 x - 4\cos x - 1 = 0$. Note that $\cos x = -\frac{1}{2}$ is a solution, and factorise as $(2\cos x + 1)(4\cos^2 x - 2\cos x - 1) = 0$. Thus $\cos x = -\frac{1}{2}$ or $\frac{2 \pm \sqrt{20}}{8}$; hence find x.

Sine and cosine laws

In any triangle with sides a, b, c and corresponding angles A, B, C we have the two important laws, the sine law,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

and the cosine law:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = c^2 + a^2 - 2ca \cos B$$
$$c^2 = b^2 + a^2 - 2ba \cos C$$

Exercises for Section 2

1. Evaluate the following summations using the formula for an arithmetic or geometric progression.

   (a) $\sum_{r=0}^{10}(2-4r)$ (b) $\sum_{k=0}^{10}8(3^k)$ (c) $\sum_{t=0}^{\infty}0.5^t$

2. Without using a calculator, evaluate or simplify the following:

   (a) $\binom{6}{3}$ (b) $\binom{40}{38}$ (c) $\frac{12!}{10!}$ (d) $\frac{(n+1)!}{(n-1)!}$ (e) $n!+(n+2)!-(n+1)!$

3. Use Pascal's triangle or the binomial theorem to expand the following fully:

   (a) $(x-4)^4$ (b) $(2x+3y)^6$

4. Find the coefficients of x^r in the following expansions.

   (a) $(1+2x)^{14}$ where $r=5$ (b) $(5-2x)^{10}$ where $r=8$

   (c) $\left(1-\frac{x}{7}\right)^5$ where $r=3$

5. Obtain the series expansion of each of the following, as far as the fourth non-zero term. State the range of validity in each case.

   (a) $(2+x)^{-1}$ (b) $(1+4x)^{-3}$ (c) $\frac{x}{(1+x)^2}$ (d) $(8-x)^{2/3}$

6. Without using a calculator, express each of the following in its simplest form:

   (a) $16^{-3/2}$ (b) $\log_2\frac{1}{8}$ (c) $\log_9 3$

   (d) $(x^3)^{-2}\times(x^{-1})^4$, where $x>0$ (e) $\sqrt{1-2x+x^2}$

   (f) $\sqrt{a}+\frac{b}{\sqrt{a}}$ (g) $\frac{1+\sqrt{2}}{1-\sqrt{2}}$ (h) $(x+y)^{\frac{1}{2}}-x(x+y)^{-\frac{1}{2}}$

7. Solve the following equations for x:

   (a) $3^{-x}=1$ (b) $e^{2x}=e$ (c) $\log_4 x=5$

   (d) $\log_x 36=2$ (e) $4^{x+2}5^{x+1}=32000$ (f) $4^x+\frac{1}{4^x}=2$

   (g) $e^{2x}-5e^x+6=0$ (h) $2^{2x}+8=9(2^x)$ (i) $\ln(\sqrt{x})=\ln x+\ln 3$

8. Expand the following in terms of $\sin a, \sin b, \cos a$ and $\cos b$:

   (a) $\sin(a+2b)$ (b) $\tan 2a$ (c) $\cos(a+b)-\cos(a-b)$ (d) $\cos 3b$

9. (a) Express $150°$ in radians. (b) Express $\frac{3\pi}{5}$ radians in degrees.

   (c) Write down the exact values of $\cos\frac{3\pi}{4}$ and $\tan\frac{7\pi}{6}$.

   (d) An arc of a circle of radius r subtends an angle θ radians at the centre.

   Write down the formulae for the perimeter and area of the sector formed.

   (e) In triangle ABC, $AB=10$ cm, $BC=8$ cm and angle $ABC=\frac{2\pi}{3}$.

   What is the exact area of the triangle?

10. Find in surd form (a) $\sin \frac{7\pi}{12}$, (b) $\tan \frac{\pi}{8}$.

(c) Without using a calculator, prove that $\cos 80° + \cos 40° = \cos 20°$.

11. Solve the following trigonometric equations

(a) $4\cos^2 x = 1$, giving the general solution. (b) $2\tan^2 x + \sec x = 1, -\pi \leq x < \pi$

(c) $\sin 2x = \tan x$, giving the general solution. (d) $\sin 3x - \sin x = \cos 2x, 0 \leq x < \pi$

3. Calculus

Later in the course we shall go into the theory of differentiation and integration in more detail. At this stage, you should just be able to use the following methods.

Differentiation

If $y = \mathrm{f}(x)$, the **derived function**, **derivative** or **differential coefficient** of y with respect to x is $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)$. This gives the **rate of change** of y with respect to x.

Graphically, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is the *gradient* of the graph of $y = \mathrm{f}(x)$ at the point (x, y), i.e. the slope of the **tangent** to the graph at this point. The **normal**, which is perpendicular to the tangent, has gradient $-\dfrac{1}{\mathrm{f}'(x)}$.
(Recall that the product of the gradients of perpendicular lines is –1.)

At a turning point or **stationary point**, $\mathrm{f}'(x) = 0$. The nature of such a point can be determined from the sign of the **second derivative** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \mathrm{f}''(x)$: this is positive at a minimum and negative at a maximum. If at a point $x = x_0$, $\mathrm{f}''(x_0) = 0$, and f'' changes sign as x passes through x_0, then f has a point of inflection at $x = x_0$.

There may also exist points of inflection which are not stationary, where $\mathrm{f}''(x) = 0$ but $\mathrm{f}'(x) \neq 0$. The tangent to a curve at any point of inflection *crosses* the curve there.

We have the following standard rules. Let u and v be functions of x.

- Sum / Difference Rule : $\dfrac{\mathrm{d}}{\mathrm{d}x}(u \pm v) = \dfrac{\mathrm{d}u}{\mathrm{d}x} \pm \dfrac{\mathrm{d}v}{\mathrm{d}x}$.
- Product Rule: To differentiate one function multiplied by another, $\dfrac{\mathrm{d}}{\mathrm{d}x}(uv) = u\dfrac{\mathrm{d}v}{\mathrm{d}x} + v\dfrac{\mathrm{d}u}{\mathrm{d}x}$.
- Quotient Rule: To differentiate one function divided by another, $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{u}{v}\right) = \dfrac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$.
- Function of a Function (Composite Function) or Chain Rule:
  To differentiate one function *of* another function, $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{f}(\mathrm{g}(x))) = \mathrm{f}'(\mathrm{g}(x)) \cdot \mathrm{g}'(x)$.
  This may be easier to use in the following form : if y is a function of u and u is a function of x, then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y}{\mathrm{d}u} \cdot \dfrac{\mathrm{d}u}{\mathrm{d}x}$.
- Standard Derivatives

$\mathrm{f}(x)$	$\mathrm{f}'(x)$	$\mathrm{f}(x)$	$\mathrm{f}'(x)$
$(ax+b)^n$	$na(ax+b)^{n-1}$	e^{ax+b}	$a\mathrm{e}^{ax+b}$
$\ln(ax+b)$	$\frac{a}{ax+b}$	$\sin(ax+b)$	$a\cos(ax+b)$
$\cos(ax+b)$	$-a\sin(ax+b)$	$\tan(ax+b)$	$a\sec^2(ax+b)$
$\cot(ax+b)$	$-a\ \mathrm{cosec}^2(ax+b)$	$\sec(ax+b)$	$a\sec(ax+b)\tan(ax+b)$
$\mathrm{cosec}(ax+b)$	$-a\ \mathrm{cosec}(ax+b)\cot(ax+b)$		

You should know these results when $a = 1, b = 0$ especially well.

Examples

1. Find the gradient of $y = \tan(2x + \pi)$ at the point $P(\pi, 0)$.
$\dfrac{\mathrm{d}y}{\mathrm{d}x} = 2\sec^2(2x+\pi) = 2\sec^2 3\pi$ at P. Now $\cos 3\pi = -1$, so the gradient is 2.

2. Find the point on $y = \sqrt{2-3x}$ where the gradient is -3.
$\dfrac{\mathrm{d}}{\mathrm{d}x}(2-3x)^{1/2} = \dfrac{1}{2}(2-3x)^{-1/2} \cdot (-3) = \dfrac{-3}{2\sqrt{2-3x}}$.
When this equals -3, $\sqrt{2-3x} = \dfrac{1}{2}$, so $2 - 3x = \dfrac{1}{4}$, so $x = \dfrac{7}{12}$. The point is $\left(\dfrac{7}{12}, \dfrac{1}{2}\right)$.

3. Find $\mathrm{f}''(x)$ when $\mathrm{f}(x) = \ln(2x+1)$.
$\mathrm{f}'(x) = \dfrac{2}{2x+1} = 2(2x+1)^{-1}$, so $\mathrm{f}''(x) = -2(2x+1)^{-2} \cdot (2) = \dfrac{-4}{(2x+1)^2}$

4. Find the turning points on the graph of $y = x^3 - 3x^2$. $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 - 6x = 0$ at turning points. $3x(x-2) = 0$, so $x = 0$ or $x = 2$. The points are $(0, 0)$ and $(2, -4)$.

5. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{3x}\sin 2x) = 2\mathrm{e}^{3x}\cos 2x + 3\mathrm{e}^{3x}\sin 2x$ [Using product rule]

6. $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln x}{x^2}\right) = \dfrac{(x^2)(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{1 - 2\ln x}{x^3}$ [Using quotient rule]

7. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sin^3(x+3)) = 3\sin^2(x+3)\cos(x+3)$ [Using composite function rule]

8. $\dfrac{\mathrm{d}}{\mathrm{d}x}(\sec(\ln x)) = \dfrac{\sec(\ln x)\tan(\ln x)}{x}$ [Using composite function rule]

Integration

Integration is the reverse of differentiation. Given a formula for the gradient or rate of change of a function, we can integrate to find the function itself.

An **indefinite** integral, written $\displaystyle\int \mathrm{f}(x)\ \mathrm{d}x$, must include an arbitrary constant.

A **definite** integral, written $\int_a^b \mathrm{f}(x)\ \mathrm{d}x$, means [$\int \mathrm{f}(x)\ \mathrm{d}x$ evaluated at $x = b$] $-$ [$\int \mathrm{f}(x)\ \mathrm{d}x$ evaluated at $x = a$]. This has a fixed numerical value, and represents the area under the graph of $y = \mathrm{f}(x)$, above the x-axis, between $x = a$ and $x = b$. Any area below the x-axis is negative, so if the graph crosses the x-axis between $x = a$ and $x = b$ the definite integral does not give the total area of the regions formed.

When the area under a graph is rotated completely about the x-axis, the volume of the solid generated is $\displaystyle\pi\int_a^b [\mathrm{f}(x)]^2\ \mathrm{d}x$.

Standard Integrals [All + c.] Again, the special cases $a = 1, b = 0$ should be familiar.

$\mathrm{f}(x)$	$\int \mathrm{f}(x)\mathrm{d}x$	
$(ax+b)^n \quad (n \neq -1)$	$\frac{1}{a}\frac{(ax+b)^{n+1}}{n+1}$	
$\frac{1}{ax+b}$	$\frac{1}{a}\ln\lvert ax+b\rvert$	
e^{ax+b}	$\frac{1}{a}\mathrm{e}^{ax+b}$	
$\sin(ax+b)$	$-\frac{1}{a}\cos(ax+b)$	
$\cos(ax+b)$	$\frac{1}{a}\sin(ax+b)$	
$\tan(ax+b)$	$\frac{1}{a}\ln\lvert\sec(ax+b)\rvert$	
$\cot(ax+b)$	$\frac{1}{a}\ln\lvert\sin(ax+b)\rvert$	
$\sec(ax+b)$	$\frac{1}{a}\ln\lvert\sec(ax+b)+\tan(ax+b)\rvert$	
$\mathrm{cosec}(ax+b)$	$\frac{1}{a}\ln\lvert\mathrm{cosec}(ax+b)-\cot(ax+b)\rvert$	
$\sec^2(ax+b)$	$\frac{1}{a}\tan(ax+b)$	(Inverse of differentiation result)
$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert$	(proved by partial fractions)
$\frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$	$\ln\lvert\mathrm{f}(x)\rvert$	(proved by substituting $u=\mathrm{f}(x)$)

Integration by Parts

$$\int u \, \mathrm{d}v = uv - \int v \, \mathrm{d}u \quad \text{or equivalently} \qquad \int u\frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

Other methods which are often useful are algebraic substitutions, partial fractions and trigonometric identities. At this stage, you will be given suitable substitutions when needed.

Examples

1. Find the area bounded by the curve $y = \mathrm{e}^{2x-1}$, the x-axis and the lines $x = 0$ and $x = 3$.
   Area $= \displaystyle\int_0^3 \mathrm{e}^{2x-1} \, \mathrm{d}x = \left[\frac{1}{2}\mathrm{e}^{2x-1}\right]_0^3 = \frac{1}{2}\left(\mathrm{e}^5 - \mathrm{e}^{-1}\right).$

2. Find the volume formed when the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated once about the x-axis.
   Volume $= \displaystyle\pi\int_0^\pi \sin^2 x \, \mathrm{d}x = \pi\int \frac{1-\cos 2x}{2} \, \mathrm{d}x = \frac{\pi}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^\pi = \frac{\pi^2}{2}.$

3. Find $\displaystyle\int x\mathrm{e}^{2x} \, \mathrm{d}x$. Integrate by parts : let $u = x$, $\mathrm{d}v = \mathrm{e}^{2x} \, \mathrm{d}x$. Then $\mathrm{d}u = \mathrm{d}x$, $v = \dfrac{1}{2}\mathrm{e}^{2x}$ so we get $\dfrac{1}{2}x\mathrm{e}^{2x} - \displaystyle\int \frac{1}{2}\mathrm{e}^{2x} \, \mathrm{d}x = \frac{1}{4}\mathrm{e}^{2x}(2x-1) + c.$

4. Using partial fractions : $\displaystyle\int_6^7 \frac{x}{x^2-x-12} \, \mathrm{d}x = \int_6^7 \frac{4}{7(x-4)} + \frac{3}{7(x+3)} \, \mathrm{d}x$
   $= \displaystyle\left[\frac{4}{7}\ln|x-4| + \frac{3}{7}\ln|x+3|\right]_6^7 = \frac{1}{7}(4\ln 3 + 3\ln 10 - 4\ln 2 - 3\ln 9) = \frac{1}{7}\ln\frac{125}{18}.$
   Check that you understand how the logs have been simplified.

5. Find $\displaystyle\int x(x^2+3)^{-1/2} \, \mathrm{d}x.$

   If you really understand the composite function rule for differentiation, you can spot the answer to this at a glance. Since x^2+3 differentiates to $2x$ and $u^{1/2}$ differentiates to $\frac{1}{2}u^{-1/2}$, the answer must be $(x^2+3)^{1/2}+c$. If you don't see this, make the substitution $u = x^2+3$, so $\mathrm{d}u = 2x \, \mathrm{d}x$, etc.

6. $\displaystyle\int x\sqrt{2x+3} \, \mathrm{d}x.$ [This is defined only if $x \geq -\frac{3}{2}$.]

   Substitute $u = 2x+3$, so $\mathrm{d}u = 2 \, \mathrm{d}x$. Then $\displaystyle\int \frac{u-3}{2}u^{1/2}\frac{1}{2} \, \mathrm{d}u = \frac{1}{4}\int u^{3/2} - 3u^{1/2} \, \mathrm{d}u$
   $= \displaystyle\frac{1}{4}\left(\frac{2}{5}u^{5/2} - 3\times\frac{2}{3}u^{3/2} + c\right) = \frac{1}{10}(2x+3)^{5/2} - \frac{1}{2}(2x+3)^{3/2} + c$

7. Find $\displaystyle\int \cos^5 x \, \mathrm{d}x.$

   $\cos^5 x = \cos^4 x \cos x = (1-\sin^2 x)^2 \cos x$, so substitute $u = \sin x, \mathrm{d}u = \cos x \, \mathrm{d}x$ to get $\displaystyle\int (1-u^2)^2 \, \mathrm{d}u = \int (1-2u^2+u^4) \, \mathrm{d}u = u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + c = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + c.$

8. $\displaystyle\int \frac{x^2}{x^3+4}\,\mathrm{d}x = \frac{1}{3}\int \frac{3x^2}{x^3+4}\,\mathrm{d}x = \frac{1}{3}\ln|x^3+4| + c.$

(Recognising an integrand of the form $\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$. Alternatively, substitute $u = x^3 + 4$.)

Exercises for Section 3

1. Differentiate each of the following with respect to x:

   (a) e^{4x+3} (b) $x^3 \cos \frac{x}{4}$ (c) $x^4 \ln 2x$

   (d) $\tan^2\left(\frac{x}{2}\right)$ (e) $\log_{10} x$ (f) $\ln(\cos(x^2))$

2. (a) Find the gradient of the curve $y = e^x \sin x$ at the point where $x = 0$.

   (b) Find the coordinates of the point on the graph of $y = \frac{1}{x^2}$ where the gradient is 2.

   (c) Find the coordinates of the three turning points on the graph of $y = 3x^4 - 4x^3 - 24x^2 + 48x$, and identify each as a maximum or minimum.

   Also find the x-coordinates of the points of inflection.

   Illustrate these features on a sketch of the graph.

   (d) Find equations of the tangent and normal to the curve $y = \sqrt{x+1}$ at $(3, 2)$.

3. Integrate each of the following with respect to x:

   (a) $\frac{5}{x^6}$ (b) $6x(x^2 - 2)^{1/2}$ (c) $-9e^{-3x} + \frac{1}{2x}$

   (d) $\frac{x}{x+2}$ (e) $\frac{3-4x}{9-x^2}$ (f) $\sec\left(\frac{3x+2}{4}\right)$

4. Evaluate the following definite integrals:

   (a) $\int_0^{4\pi} \sin(3x - 4\pi)\ dx$ (b) $\int_1^2 (x+1)^4\ dx$

   (c) $\int_0^3 \frac{2x+1}{x^2+x+2}\ dx$ (d) $\int_1^e \frac{1}{x(1+\ln x)}\ dx$

5. Find (a) the area bounded by $y = e^{2x+3}$, the x and y axes and the line $x = -2$,

   (b) the volume formed when the area under $y = \sec x$ between $x = 0$ and $x = \frac{\pi}{4}$ is rotated once about the x-axis.

6. Evaluate the following using the given substitution (or otherwise):

   (a) $\int x\sqrt{x^2 - 1}\ dx \quad (u = x^2 - 1)$ (b) $\int_0^{\sqrt{\pi}} 3x \sin(x^2)\ dx \quad (u = x^2)$

   (c) $\int \cos^3 x\ dx \quad (u = \sin x)$ (d) $\int_0^{\pi/2} \cos^4 x \sin x\ dx \quad (u = \cos x)$

7. Use integration by parts to find:

   (a) $\int x \sec^2 x\ dx$ (b) $\int x^2 \ln x\ dx$ (c) $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} x\,\mathrm{cosec}^2 x\ dx$

8. Using the Partial Fractions which you may already have found in an earlier exercise, integrate each of the following with respect to x:

   (a) $\frac{5x+1}{x^2+x-2}$ (b) $\frac{x^2+31x}{(x+1)(x-4)(2x-1)}$ (c) $\frac{x^3-1}{(x-2)(x-3)}$

   (d) If the area under the curve $y = \frac{x+1}{3x^2+2x}$ between $x = 1$ and $x = 3$ is $\frac{1}{6} \ln k$, find the exact value of k.

Longer Questions

1. Find the set of real values of k, $k \neq -1$, for which the roots of the equation $x^2 + 4x - 1 + k(x^2 + 2x + 1)$ are real and distinct.

2. Find constants a and b such that when $x^5 - ax^3 - bx^2$ is divided by $x^2 - x - 2$, the remainder is $x + 2$.

3. Express $6x^2 - 25x - \dfrac{25}{x} + \dfrac{6}{x^2}$ in terms of y, where $y = x + \dfrac{1}{x}$.
   Hence solve the equation $6x^4 - 25x^3 + 38x^2 - 25x + 6 = 0$.

4. If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, find expressions in terms of a, b, c, d for $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

5. The arithmetic mean and the geometric mean of the positive real numbers x and y are respectively $A = \dfrac{x+y}{2}$ and $G = \sqrt{xy}$. Prove that $A > G$. [Hint : consider $A - G$.]

6. The second and fifth terms of a geometric progression of positive terms are $\sqrt{2}$ and $8\sqrt{2}$ respectively. Find the exact value of the sum of the first nine terms.

7. Given that $\mathrm{f}(x) \equiv \dfrac{x+1}{(x-1)(2x+1)}$, find an expression for $\mathrm{f}(x)$ as a series in ascending powers of x up to and including the term in x^5.

8. Given that $x > 2$, find the constants a, b, c for which
   $$\left(\frac{x+2}{x}\right)^{-1/2} = a + \frac{b}{x} + \frac{c}{x^2} + \frac{d}{x^3} + \cdots$$
   Use this series with a suitable value of x to find an approximate value for $\left(\dfrac{450}{51}\right)^{1/2}$, to 4 decimal places.

9. If a, b, x are positive and neither a nor b is 1, prove that $\log_b x = \dfrac{\log_a x}{\log_a b}$

10. Sketch on the same diagram the graphs of $y = \dfrac{2}{3}\cos x$ and $y = \tan x$ for $0 \leq x \leq 2\pi$. Calculate the values of x at the points of intersection of the two graphs. Hence state the set of values of x in the given interval for which $2\cos x > 3\tan x$.

11. $ABCD$ is a square field and O is the mid-point of AB. A goat is tethered to O and can graze exactly half the field. If L, M are the furthest points which the goat can reach on AD, BC respectively, and angle $LOM = \alpha$ radians, prove that $\dfrac{\alpha}{2} = 1 - \cos\alpha - \dfrac{1}{2}\sin\alpha$.

12. Starting from the formulae for $\sin(x+y)$ and $\cos(x+y)$, prove the formula for $\tan(x+y)$. Use this result to find, to 2 decimal places, the positive value of x for which $\arctan x + \arctan 2x = \dfrac{\pi}{4}$. [arctan x is the inverse tangent $\tan^{-1} x$.]

13. Find the values of x, in terms of p, which satisfy the equation $x^3 - 7p^2x + 6p^3 = 0$.
    Hence find the values of t, in radians to 2 decimal places in the interval $0 \leq t < 2\pi$, for which $4\sec^3 t - 7\sec t + 3 = 0$.

14. Given that $x = 2 + 2\mathrm{e}^{-2t} - \mathrm{e}^{-3t}$, find the maximum value of x as t varies.

15. Find the turning points of the curve $y = x^3 - x$ and decide their nature. Find also the coordinates of the point of inflection of the curve. Sketch the curve.

    Find the equation of the tangent to the curve at (1, 0) and find also the coordinates of the other point at which this tangent cuts the curve.

16. Find the maximum and minimum turning points of $y = x + \dfrac{1}{x+1}$. Sketch the graph of this function.

17. Find the values of x for which the function $\mathrm{e}^{-x/a}\sin x$, where a is a positive constant, has maximum and minimum values. Show that the sequence of maximum and minimum values of the function forms a geometric progression and state its common ratio.

18. Using the substitution $u = \sin x$, evaluate to two decimal places the integral

$$\int_{\pi/6}^{\pi/2} \frac{4\cos x}{3+\cos^2 x}\,\mathrm{d}x.$$

19. Find the area bounded by the curve $y = \dfrac{x+1}{\sqrt{x+2}}$, the lines $x = 1$ and $x = 2$ and the x-axis. Also find the volume formed when this area is rotated once about the x-axis.

20. Use the substitution $u = \dfrac{\pi}{2} - x$ to show that if

$$I = \int_0^{\pi/2} \frac{2\cos x + 7\sin x}{\cos x + \sin x}\,\mathrm{d}x \qquad \text{and} \qquad J = \int_0^{\pi/2} \frac{2\sin x + 7\cos x}{\cos x + \sin x}\,\mathrm{d}x,$$

    then $I = J$.

    By considering $I + J$, find the value of I.

Answers to Exercises

Please let me know if you find any mistakes.

1. Algebraic Methods

1. (a) $5x - 20y - 60z + 360$
   (b) xy
   (c) $x^3 - y^3$
   (d) $3xy - 3xz + yz$
2. (a) $\frac{x^4}{y^3}$
   (b) $\frac{1}{2x+3y}$
   (c) $\frac{4y+4}{y}$
   (d) $\frac{9x-2}{18}$ or $\frac{x}{2} - \frac{1}{9}$
   (e) $\frac{2x^2+2x+1}{x^2+x}$
   (f) $\frac{3-x}{x^2-1}$
3. (a) $(x-4)^2 - 23$
   Min. -23 when $x = 4$
   (b) $\frac{45}{4} - (2x - \frac{1}{2})^2$
   Max. $\frac{45}{4}$ when $x = \frac{1}{4}$
   (c) $(x^2+2)^2 + 3$
   Min. 7 when $x = 0$
4. (a) $(x+2)(x+3)$
   (b) $(x-2)(x+2)(x^2+3)$
   (c) $2(a-3x)(2x-3y)$
   (d) $(2x+1)(2x+3)$
   (e) $(x-1)(x+2)(x-3)$
   (f) $(11p-13q)(11p+13q)$
   (g) $(x+4y)(x^2-4xy+16y^2)$
   (h) $x^2(x+4)(x-4)$
   (i) $12x(x-3y)$
   (j) 0
5. (a) $x - 5$
   (b) $x + \frac{1}{x+1}$
   (c) $x^2 - 2x + 1 - \frac{5}{2x-3}$
   (d) $x + 1 + \frac{4-x}{x^2+x+1}$
6. (a) $\frac{2}{x-1} + \frac{3}{x+2}$
   (b) $\frac{4}{x-4} - \frac{2}{x+1} - \frac{3}{2x-1}$
   (c) $x + 5 + \frac{26}{x-3} - \frac{7}{x-2}$
7. (a) Show f(1/2) = 0
   (b) f(–1) = 1
   (c) $a = 0, b = -6$
8. (a) $x = -35/2$
   (b) $x = 1/2, y = -2$
   (c) $x = 4 \pm \sqrt{29}$
   (d) $x = 1, x = 2, x = -3$
   (e) $(1,7), (-0.98, -7.14), (49.98, 0.14)$
9. (a) $0 < k < 1$
   (b) Sum = $2p$, product = $-3p^2$
10. (a) $x \geq -1/3$
    (b) $x < \frac{1}{2}$ or $x > 3$
    (c) $x \leq -5$ or $x \geq 3$
    (d) $x < -1$ or $x > -0.5$
    (e) $-1 < x < 7$
11. (a) $x = \frac{d-b}{a-c}$
    (b) $x = 0, x = 3p$
    (c) $x = 0, x = \frac{k-1}{k}$
    (d) $x = 2y \pm z$
12. (a) a^2
    (b) $\frac{1}{2}$
    (c) $1 - 2q^2$
    (d) 13

2. Series, Functions, etc.

1. (a) -198
   (b) 708584
   (c) 2
2. (a) 20
   (b) 780
   (c) 132
   (d) $n(n+1)$
   (e) $n!(n^2+2n+2)$
3. (a) $x^4 - 16x^3 + 96x^2 - 256x + 256$
   (b) $64x^6 + 576x^5y + 2160x^4y^2 + 4320x^3y^3 + 4860x^2y^4 + 2916xy^5 + 729y^6$

4. (a) 64064
   (b) 288000
   (c) $-10/343$

5. (a) $\frac{1}{2} - \frac{1}{4}x + \frac{1}{8}x^2 - \frac{1}{16}x^3, |x| < 2$
   (b) $1 - 12\,x + 96\,x^2 - 640\,x^3, |x| < 1/4$
   (c) $x - 2\,x^2 + 3\,x^3 - 4\,x^4, |x| < 1$
   (d) $4 - \frac{1}{3}x - \frac{1}{144}x^2 - \frac{1}{2592}x^3, |x| < 8$

6. (a) $\frac{1}{64}$
   (b) -3
   (c) $\frac{1}{2}$
   (d) x^{-10}
   (e) $x - 1$ if $x \geq 1$, $\quad 1 - x$ if $x \leq 1$
   (f) $\frac{a+b}{\sqrt{a}}$
   (g) $-3 - 2\sqrt{2}$
   (h) $\frac{y}{(x+y)^{\frac{1}{2}}}$ or $\frac{y}{\sqrt{x+y}}$

7. (a) 0
   (b) $\frac{1}{2}$
   (c) 1024
   (d) 6
   (e) 2
   (f) 0
   (g) $\ln 2$, $\ln 3$
   (h) 0, 3
   (i) $\frac{1}{9}$

8. (a) $2\sin a\cos^2 b - \sin a + 2\,\cos a\sin b\cos b$
   (b) $\frac{2\sin a\cos a}{2\cos^2 a - 1}$
   (c) $-2\sin a\sin b$
   (d) $4\cos^3 b - 3\cos b$

9. (a) $\frac{5\pi}{6}$
   (b) $108°$
   (c) $\frac{-1}{\sqrt{2}}, \ \frac{1}{\sqrt{3}}$
   (d) $r\theta + 2r, \ \frac{1}{2}r^2\theta$
   (e) $20\sqrt{3}$ cm^2

10. (a) $\frac{\sqrt{6}+\sqrt{2}}{4}$
    (b) $\sqrt{2} - 1$
    (c) By $\cos x + \cos y$ formula,
    $\cos 80° + \cos 40° = 2\cos 60°\cos 20°$
    $= 2(\frac{1}{2})\cos 20° = \cos 20°$

11. (a) $n\pi \pm \pi/3$
    (b) $-2.3, 0, 2.3$
    (c) $n\pi, \quad n\pi \pm \pi/4$
    (d) $\pi/6, \pi/4, 3\pi/4, 5\pi/6$

3. Calculus

1. (a) $4e^{4x+3}$
   (b) $3x^2\cos\frac{x}{4} - \frac{x^3}{4}\sin\frac{x}{4}$
   (c) $x^3(1 + 4\ln 2x)$
   (d) $\tan\frac{x}{2}\sec^2\frac{x}{2}$
   (e) $\frac{1}{x\ln 10}$
   (f) $-2x\tan(x^2)$

2. (a) 1
   (b) $(-1, 1)$
   (c) $(1, 23)$ max, $(-2, -112)$ and $(2, 16)$ min. Inflexion points when $x = \frac{1}{3}(1 \pm \sqrt{13})$
   (d) $4y - x = 5, \ y + 4x = 14$

3. (a) $-\frac{1}{x^5} + c$
   (b) $2(x^2 - 3)^{3/2} + c$
   (c) $3e^{-3x} + \frac{1}{2}\ln|x| + c$
   (d) $x - 2\ln|x + 2| + c$
   (e) $\frac{3}{2}\ln|x - 3| + \frac{5}{2}\ln|x + 3| + c$
   (f) $\frac{4}{3}\ln\left|\sec\left(\frac{3x+2}{4}\right) + \tan\left(\frac{3x+2}{4}\right)\right| + c$

4. (a) 0
   (b) $211/5$
   (c) $\ln 7$
   (d) $\ln 2$

5. (a) $\frac{1}{2}(e^3 - e^{-1})$
   (b) π

6. (a) $\frac{1}{3}(x^2 - 1)^{3/2} + c$
   (b) 3
   (c) $\sin x - \frac{1}{3}\sin^3 x + c$
   (d) $\frac{1}{5}$

7. (a) $x\tan x - \ln|\sec x| + c$
   (b) $\frac{x^3}{9}(3\ln x - 1) + c$
   (c) π

8. (a) $2\ln|x-1| + 3\ln|x+2| + c$
   (b) $4\ln|x-4| - \frac{3}{2}\ln|2x-1|$
   $- 2\ln|x+1| + c$
   (c) $\frac{1}{2}x^2 + 5x + 26\ln|x-3|$
   $- 7\ln|x-2| + c$
   (d) $\frac{135}{11}$

18. 0.59

19. $4/3, \quad \pi(3/2 + \ln(4/3))$

20. $9\pi/4$

Longer Questions

1. $k > -5/4$
2. $a = 3, b = 1$
3. $6y^2 - 25y - 12 \qquad x = 2/3, 1, 3/2$
4. $-b/a, c/a, -d/a$
5. Show that $A - G = (\sqrt{x} - \sqrt{y})^2/2$, etc.
6. $511\sqrt{2}/2$
7. $-1 - 2\,x^2 + 2\,x^3 - 6\,x^4 + 10\,x^5$
8. $a = 1, b = -1, c = 3/2, d = -5/2$.
   Put $x = 100$ (why?), get 2.9704
9. Let $y = \log_b x$, so $x = b^y$. Take logs to base a of both sides.
10. Intersect at $x = \pi/6, 5\pi/6$.
    $0 \le x < \pi/6, \quad \pi/2 < x < 5\pi/6,$
    $3\pi/2 < x \le 2\pi$
11. Let side of field $= 2a$ and find radius of sector in terms of a ...
12. $\frac{\sqrt{17}-3}{4}$
13. $x = p, 2p, -3p \qquad t = 0, 2.30, 3.98$
14. 86/27
15. $(1/\sqrt{3}, -2/3\sqrt{3})$ minimum,
    $(-1/\sqrt{3}, 2/3\sqrt{3})$ maximum.
    Point of inflection (0, 0).
    $y = 2x - 2 \qquad (-2, -6)$
16. (0, 1), $(-2, -3)$.
    Curve with asymptotes $x = -1, y = x$
17. Max / min when $\tan x = a$.
    Ratio $= -\exp(\pi/a)$