CM 1021 Mathematical Methods for Computing I

Exercise Sheet 8 - Solutions

1. (a) A sum of £10,000 is invested in an account that pays 5% per annum simple interest. How much money is in the account after 10 years?

$$\pounds 10,000 + (10,000 \times 0.05 \times 10) = \pounds 15,000$$

(b) A sum of £10,000 is invested in an account that pays 5% per annum compound interest. How much money is in the account after 10 years?

$$\pounds 10,000 \times (1.05)^{10} = \pounds 16,288.95$$

(c) A sum of £10,000 is invested in an account that pays simple interest. What interest rate is required so that the money in the account after 10 years is the same as if the money were invested with 5% compound interest (as in part (b)).

$$\frac{16,288.95 - 10,000}{10 \times 10000} \times 100\% = 6.2889\%$$

(d) A sum of £10,000 is invested in an account that pays 5% per annum compounded continuously. How long will it take for the amount in the account to reach £15000?

$$10000\,\mathrm{e}^{0.05\,t} = 15000 \quad \Rightarrow \quad t = 8.11 \quad \text{years}$$

(e) If £10,000 is invested, at what interest rate, compounded monthly will the amount in the account total £12500 after 5 years?

$$10000(1+r)^6 0 = 12500 \quad \Rightarrow \quad r = 0.00373\% \quad \text{monthly} \quad = 4.47 \quad \text{annually}\%$$

2. A sum of £25 per month is paid into a savings account which offers a return of 3% per annum, compounded monthly.

(a) How much money will be in the account after 10 years?

$$\frac{25\left((1+\frac{0.03}{12})^{120} - 1\right)(1+\frac{0.03}{12})}{\frac{0.03}{12}} = \pounds 3,502.27$$

(b) What sum should be invested each month in order to accumulate a total of £4000 in the account after 10 years?

$$\frac{4000}{3502.27} \times 25 = \pounds 28.55$$

(c) What interest rate is needed if £25 is invested monthly and the total in the account is to reach £4500 after 10 years?

$$25\frac{(((1+r)^{120}-1)(1+r))}{r} \qquad \Rightarrow r = 0.00633\% \quad \text{monthly} \quad = 7.59 \quad \text{annually}\%$$

3. In order to buy a house, a person takes out a mortgage of £110,000. The interest rate is fixed at 8% per annum.

(a) Calculate the monthly payments if the mortgage is to be paid off in 25 years.

$$P_{25} = \frac{\frac{0.08}{12} \times 110,000 \times \left(1+\frac{0.08}{12}\right)^{300}}{\left(1+\frac{0.08}{12}\right)^{300}-1} = £849$$

(b) Calculate the total amount of interest paid by the person over the 25 year loan period.

$$300 \times 848.998 - 110,000 = £144,699$$

(c) If the original £110,000 mortgage was paid over a period of 20 years, how much money would have been saved compared with paying over a 25 year period?

$$P_{20} = \frac{\frac{0.08}{12} \times 110,000 \times \left(1+\frac{0.08}{12}\right)^{240}}{\left(1+\frac{0.08}{12}\right)^{240}-1} = £920.08$$

additional interest is $300 \times P_{25} - 240 \times P_{20} = £33,879.17$.

(d) What is the loan outstanding after 15 years?

$$V_{180} = 110000\left(1+\frac{0.08}{12}\right)^{180} - \frac{849\left(\left(1+\frac{0.08}{12}\right)^{180}-1\right)}{1+\frac{0.08}{12}} = 69,97760$$

The amount of the loan that has been repaid is thus $110,000 - 69976 = £40,024$

(e) If the house initially cost £140,000 and house prices rise at an average 4% per year, how much equity will the person have after 15 years?

The value of the house after 15 years will have risen to $140,000(1.04)^{15} = £252,132$. Subtracting the amount of the loan outstanding after 15 years, the equity in the house is $182,156$

4. How long will it take for an investment to triple in value if invested at an annual rate of 3% compounded (a) annually and (b) continuously?

$$(a) \quad 1.03^n = 3, \quad n = 37.16\text{years}$$

$$(b) \quad e^{0.03t} = 2, \quad t = 36.62\text{years}$$

5. What is the APR if the nominal rate of 7% is compounded (a) quarterly and (b) continuously?

$$(a) \quad 100 \times \left(\left(1 + \frac{0.07}{4} \right)^4 - 1 \right) = 7.1859\%$$

$$(b) \quad 100 \times \left(e^{0.07} - 1 \right) = 7.2508$$

6. What is the annual percentage rate if the nominal rate is 12% compounded weekly?

$$100 \times \left(\left(1 + \frac{0.12}{52} \right)^{52} - 1 \right) = 12.734\%$$

7. Two projects are available for an initial investment of £100,000.

   Project A returns £25,000 at the end of year 2 and £27,500 at the end of years 3,4,5 and 6. At the end of year 7 the scrap value will be £20,000.

   Project B gives a return of £28,000 at the end of years 1,2,3 4 and 5.

   (a) Calculate the NPV and the IRR of each project.

   For project A

$$NPV_A = \frac{25,000}{1.04^2} + \sum_{n=3}^{6} 27,500 \left(1.04^n\right)^{-1} - \frac{20,000}{1.04^7} - 100,000 = 30,603$$

$$25,000 \left(1 + r\right)^{-2} + \sum_{n=3}^{6} 27,500 \left(\left(1 + r\right)^n\right)^{-1} + 20000 \left(1 + r\right)^{-7} = 100,000 \qquad \text{thus} \quad r = 10.78\%$$

   For project B

$$NPV_B = \sum_{n=1}^{5} 28,000 \left(1.04^n\right)^{-1} - 100,000 = 24,651$$

$$\sum_{n=1}^{5} 28,000 \left(\left(1 + r\right)^n\right)^{-1} = 100,000 \qquad \text{thus} \quad r = 12.38$$

   (b) Which project is the more attractive investment?

   Project A has a higher NPV, but a lower IRR than project B. We would probably prefer project A because of its higher present value but might be concerned that realising the projected scrap value in 7 years time is an important part of the project profitability. Without the scrap value to NPV falls to £15405.

More Challenging Question

8. A prize fund is set up with a single investment of £5000 to provide an annual prize of £500. The fund is invested to earn 7% interest compounded annually. If the first time the prize is awarded is one year after the initial investment is made and subsequent prizes are awarded each year on the same date, for how many years can the prize be awarded before the fund falls below £500?

The simplest way to solve this is to write a simple loop in Maple such as

$>\quad a_1 = 5000 * 1.07;$
$>$ for n from 2 to 17 while $a_n > 500$ do:
$> a_n := (a_{n-1} - 500) * 1.07;$
$> a_n$

This will compute the amount in the fund at the end of each year. At the end of year 17 there is only £400.15 in the fund and so no more prizes can be awarded after the start of year 17.