CM 1021 Mathematical Methods for Computing I

Exercise Sheet 4a

1. Let $\mathbf{u} = (-3, 2, 1, 0)$, $\mathbf{v} = (4, 7, -3, 2)$, $\mathbf{w} = (5, -2, 8, 1)$, $\mathbf{x} = (1, 2, 3, 4)$

   (a) Find (i) $\mathbf{v} - \mathbf{w}$ (ii) $3\mathbf{w} + 2\mathbf{x} - \mathbf{u}$ (iii) $(6\mathbf{w} + 2\mathbf{v}) - (4\mathbf{x} - \mathbf{u})$

   (b) Find (i) $||\mathbf{u}||$ (ii) $||\mathbf{w}+\mathbf{x}||$ (iii) $||\mathbf{x}-\mathbf{u}||$ (iv) $\hat{\mathbf{w}}$ (v) $\widehat{2\mathbf{v} - \mathbf{x}}$

   (c) Find (i) $\mathbf{u} \cdot \mathbf{v}$ (ii) $\mathbf{w} \cdot \mathbf{x}$ (iii) $d(\mathbf{w}, \mathbf{v})$ (iv) $d(\mathbf{u}, \mathbf{x})$

   (d) Find a vector $\mathbf{y}$ such that $5\mathbf{y} - 2\mathbf{v} = 2(\mathbf{w} - 5\mathbf{x})$

   (e) Find the value of all scalars k such that $||k\mathbf{u}|| = 14$

   (f) Are $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ linearly independent?

2. Which of the following pairs of vectors are orthogonal?

   (a) $\mathbf{u} = (-1, 3, 2)$ $\mathbf{v} = (4, 2, -1)$

   (b) $\mathbf{u} = (-2, -2, -2)$ $\mathbf{v} = (1, 1, 1)$

   (c) $\mathbf{u} = (-4, 6, 10, 2)$ $\mathbf{v} = (2, 1, -2, 11)$

3. For what value of k are $\mathbf{u}$ and $\mathbf{v}$ orthogonal?

   (a) $\mathbf{u} = (2, 1, 3)$ $\mathbf{v} = (1, 7, k)$

   (b) $\mathbf{u} = (k, k, 1)$ $\mathbf{v} = (k, 5, 6)$

4. Which of the following are linear combinations of $\mathbf{u} = (1, -2, 3)$ and $\mathbf{v} = (2, -1, 4)$

   (a) $(-4, -1, 6)$

   (b) $(0, -3, 2)$

   (c) $(4, -3, 8)$

   (d) $(3 - 5, 9)$

5. Express the following as linear combinations of
   $\mathbf{u} = (2, 1, 4), \mathbf{v} = (1, -1, 3)$ and $\mathbf{w} = (3, 2, 5)$

   (a) $(-9, -7, -15)$

   (b) $(6, 11, 6)$

   (c) $(1, 1, 1)$

   (d) $(7, 8, 9)$

6. Determine whether or not these sets of vectors span $\mathbb{R}^3$:

   (a) $(2,2,2),(0,0,3),(0,1,1)$

   (b) $(2,-1,3),(4,1,2),(8,-1,8)$

   (c) $(3,1,4),(2,-3,5),(5,-2,9)$

7. Show that the following sets of vectors are linearly independent.

   (a) $(3,8,7,-3),(1,5,3,-1),(2,-1,2,6),(1,4,0,3)$

   (b) $(0,0,2,2),(3,3,0,0),(1,1,0,1),(2,3,1,4)$

   (c) $(0,3,-3,-6),(-2,0,0,6),(0,-4,-2,-2),(0,-8,4,-4)$

   (d) $(3,0,-3,-6),(0,2,3,1),(0,-2,-2,0),(-2,1,2,1)$

More challenging questions

8. Prove that for any vectors $\mathbf{u},\mathbf{v},\mathbf{w}$ the vectors $\mathbf{u}-\mathbf{v}$, $\mathbf{v}-\mathbf{w}$ and $\mathbf{w}-\mathbf{u}$ form a linearly dependent set.

9. Find the values of λ such that the following vectors form an independent set in $\mathbb{R}^3$

$$v_1 = \left(\lambda, -\frac{1}{2}, -\frac{1}{2}\right), \quad v_2 = \left(-\frac{1}{2}, \lambda, -\frac{1}{2}\right), \quad v_3 = \left(-\frac{1}{2}, -\frac{1}{2}, \lambda\right)$$

10. What geometric properties must a set of three vectors have if they are to span $\mathbb{R}^3$?

11. Determine whether the two lines

$$r = (3,2,3,-1) + t(4,6,4,-2), \quad r = (0,3,4,5) + t(1,-3,-4,-2)$$

    intersect in $\mathbb{R}^4$