MAT 1015 Techniques in Calculus I Autumn 2009

Coursework 1 Solutions

1. By making the substitution $y = 3^x$ or otherwise, solve the equation

$$3^{2x} - 2 \times 3^{x+1} + 5 = 0.$$

Give your answer correct to 2 d.p.s.

Let $y = 3^x$. Then $y^2 - 6y + 5 = 0 \Rightarrow (y-5)(y-1) = 0 \Rightarrow y = 1, 5$. If $3^x = 5$ then $x = \frac{\ln 5}{\ln 3} = 1.46$ while if $y = 1$ then $x = 0$.

2. Solve the equation

$$2\cos x(2\cos^2 x - 1) + 1 = 2\sin^2 x,$$

(i) for $x \in [0, 2\pi]$ and

(ii) give the general solution for any x.

$$4\cos^3 x - 2\cos x + 1 = 2 - 2\cos^2 x$$
$$(2\cos x + 1)(2\cos^2 x - 1) = 0$$

Either $\cos x = -\dfrac{1}{2}, x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}$ i.e. $\pm\dfrac{2\pi}{3} + 2n\pi, n \in \mathbb{Z}$

Or $\cos x = \pm\dfrac{1}{\sqrt{2}}, x = \dfrac{\pi}{4}, \dfrac{7\pi}{4}$ i.e $\pm\dfrac{\pi}{4} + n\pi, n \in \mathbb{Z}$

3. Find the set of real values of c for which the equation

$$\frac{x^2+1}{x^2-x+1} = c$$

is satisfied by at least one real value of x.

$$x^2(1-c) + cx + 1 - c = 0$$

The discriminant $\Delta = c^2 - 4(1-c)^2 = (3c-2)(2-c)$

We need $\Delta \geq 0$. The critical points are $c = \frac{2}{3}, c = 2$ so we need $c \in \left[\frac{2}{3}, 2\right]$

4. Given that $f : x \mapsto 1 + [x]$ and $g : x \mapsto \cos\left(\frac{\pi}{x}\right)$, find $f \circ g(4)$ and $g \circ f(\pi)$, giving your answer to 2 d.p.s.

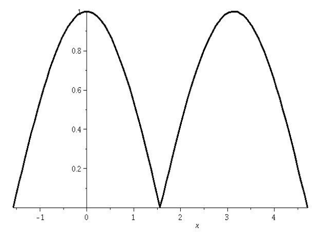
$f \circ g = 1 + [\cos\left(\frac{\pi}{x}\right)] = 1$ for $x \neq 0$

$g \circ f = \cos\left(\frac{\pi}{1+[x]}\right), \quad g \circ f(\pi) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

5. For the following, sketch the graph and state with, justification whether the functions are odd, even, periodic (state the period), injective, surjective or bijective

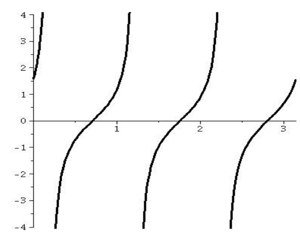
   (a) $f : x \mapsto |\cos x|$

   Not injective, surjective or bijective. Even. Periodic with period π.

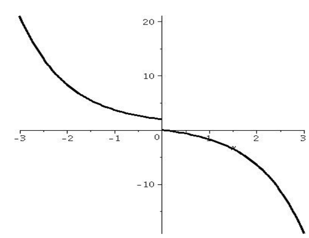
   (b) $g : x \mapsto \tan 3x + 1$

   Not odd or even, $\frac{\pi}{3}$ periodic, surjective, not injective

   (c) $h : x \mapsto 1 - \mathrm{sgn\,x\,e}^{|x|}$.

   Not odd nor even. Not injective, surjective or bijective. Not periodic.

6. For all real x define

$$f : x \mapsto -3 + 5x, \quad g : x \mapsto x^2 + 5x + 3.$$

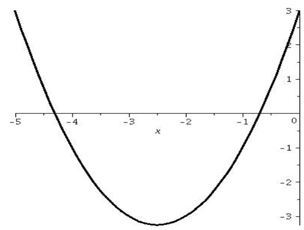
   (a) Find the range of each of f and g.
   (b) Define the inverse function f^{-1}.
   (c) Explain why g has no inverse. Restrict the domain and codomain so that $x \mapsto x^2 - 5x + 3$ is invertible and find the inverse function. How would you prove that your inverse function is correct?
   (d) Solve the equation $f \circ g = g \circ f$.

(a) Range of f is $\mathbb{R}$.

$$g(x) = \left(x + \tfrac{5}{2}\right)^2 - \frac{13}{4} \text{ Range is } \left[-\tfrac{13}{4}, \infty\right)$$

(b) $f^{-1}(x) = \dfrac{x+3}{5}$

(c) g is not bijective



We restrict the range to $\left[-\tfrac{5}{2}, \infty\right)$, codomain to $\left[-\tfrac{13}{4}, \infty\right)$

$$x = y^2 + 5y + 3 \Rightarrow y = \frac{-5 + \sqrt{25 + 4(3-x)}}{2} \Rightarrow g^{-1}(x) = \frac{1}{2}\left(-5 + \sqrt{13 - 4x}\right)$$

To show the solution is correct, show that $g(g^{-1}(x))) = 1$.

(d)

$$-3+5(x^2+5x+3) = (-3+5x)^2+5(-3+5x)+3 \Rightarrow 4x^2-6x-3 = 0 \Rightarrow x = \frac{3}{4} \pm \frac{\sqrt{21}}{4}$$

7. By substituting $x = z - \dfrac{p}{3z}$ into $x^3 + px + q = 0$ find a quadratic equation in z^3. Hence, by making a suitable substitution, transform the equation $x^3 - 30x - 36 = 0$ into a quadratic equation in z^3 and show that $z^3 = 18 \pm 26i$

   (a) Show that $(3+i)^3 = 18 + 26i$

   (b) Deduce one root of $x^3 - 30x - 36 = 0$ and find the other two roots of the equation.

$$z^6 - 1/27\,p^3 + qz^3$$

$$z^6 + 1000 - 36\,z^3 \Rightarrow y^2 - 36y + 1000 \Rightarrow y = 18 \pm 26\,i$$

Since $(3+i)^3 = 18 + 26i$ it follows that $x = 3 + i + \dfrac{10}{3+i} = 6$ so $x = 6$ is a root and thus $x - 6$ is a factor of the original equation. (By long division if necessary)

$$x^3 - 30x - 36 = (x-6)(x^2 + 6\,x + 6) \Rightarrow x = 6, -3 \pm \sqrt{3}$$

8. If $w = 2 - 3i$ and $z = -4 + 5i$, express the following complex numbers in the form $a + bi$ where a and b are real numbers. For each answer also find the modulus and the argument, in radians, between $-\pi$ and π, correct to 2 d.p.s.

(a) $w - z$ (b) $w + z$ (c) wz (d) z^2 (e) $\dfrac{z}{w}$ (f) $\dfrac{1}{z^2}$

(a) $6 - 8i, \quad 10, \quad -\tan^{-1}\dfrac{4}{3} = -0.93$

(b) $-2 + 2i, \quad 2\sqrt{2}, \quad \dfrac{3\pi}{4} = 2.36$

(c) $7 + 22i, \quad \sqrt{533}, \quad \tan^{-1}\dfrac{22}{7} = 1.27$

(d) $-9 - 40i, \quad 41 \quad \tan^{-1}\dfrac{40}{9} - \pi = -1.79$

(e) $\dfrac{23}{13} - \dfrac{2}{13}i, \quad \dfrac{\sqrt{533}}{13}, \quad \tan^{-1}\dfrac{2}{23} - \pi = -3.05$

(f) $-\dfrac{9}{1681} + \dfrac{40}{1681}i, \quad \dfrac{1}{41}, \quad \tan^{-1}\dfrac{40}{9} + \pi = 1.79$

9. z_1 is the complex number with modulus 2 and argument $\dfrac{-3\pi}{4}$.

   z_2 is the complex number with modulus $2\sqrt{2}$ and argument $\dfrac{2\pi}{3}$

   (a) Find the modulus and argument (in radians between $-\pi$ and π) of

   (i) $z_1 z_2$ (ii) $\dfrac{z_1}{z_2}$ (iii) $\dfrac{1}{z_1}$

   (i) $|z_1 z_2| = 4\sqrt{2}, \quad \arg(z_1 z_2) = -\dfrac{\pi}{12}$

   (ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{1}{\sqrt{2}}, \quad \arg\left(\dfrac{z_1}{z_2}\right) = \dfrac{7\pi}{12}$

   (iii) $\left|\dfrac{1}{z_1}\right| = \dfrac{1}{2}, \quad \arg\left(\dfrac{1}{z_1}\right) = \dfrac{3\pi}{4}$

   (b) Express each of z_1 and z_2 in the form $a + bi$ where a and b are real.

   $z_1 = -\sqrt{2} - \sqrt{2}i, \qquad z_2 = -\sqrt{2} + \sqrt{2}\sqrt{3}i$

10. Given that a and b are real numbers, prove that ai is a root of the equation

$$z^3 - bz^2 + a^2 z - a^2 b = 0$$

    If ai is a root then so is $-ai$. Thus we have factors $(z - ai)(z + ai) = z^2 + a^2$. Then we can factor the equation to get $(z - b)(z^2 + a^2) = 0$. Alternatively just make the substitution.

11. * Show that the area of a triangle with sides a, b, c is given by $\sqrt{s(s-a)(s-b)(s-c)}$, where s is the semi-perimeter of the triangle, $s = \dfrac{1}{2}(a + b + c)$

    The formula is credited to Heron of Alexandria, c. 60A.D. It has been suggested that

Archimedes knew the formula in 21 B.C. This proof is not Heron's original proof. There are other proofs which may be quicker

Area $= \dfrac{1}{2} bc \sin A$

But $\sin A = 2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$ so that we have Area$=bc\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

Also $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$

Now, since $\cos A = 2\cos^2\dfrac{A}{2} - 1$ we have

$$2\cos^2\frac{A}{2} = 1 + \frac{b^2+c^2-a^2}{2bc} = \frac{(b+c)^2-a^2}{2bc} = \frac{(b+c-a)(b-c+a)}{2bc}$$

If $s = \dfrac{a+b+c}{2}$ then $\cos^2\dfrac{A}{2} = \dfrac{(s-a)s}{bc}$

We also have $\cos A = 1 - 2\sin^2\dfrac{A}{2}$ so that

$$2\sin^2\frac{A}{2} = 1 - \frac{b^2+c^2-a^2}{2bc} = \frac{-(b-c)^2-a^2}{2bc} = \frac{(a-b+c)(a+b-c))}{2bc}$$

Once more using the expression for the semi perimeter we have

$$\sin^2\frac{A}{2} = \frac{(s-b)(s-c)}{bc}$$

If we now substitute these expressions for $\sin\dfrac{A}{2}$ and $\cos\dfrac{A}{2}$ we obtain

Area$= bc\sqrt{\dfrac{(s-b)(s-c)}{bc}\dfrac{(s-a)s}{bc}} = \sqrt{s(s-a)(s-b)(s-c)}$