UNIVERSITY OF SURREY©

B. Sc. Undergraduate Programmes in Computing

Level HE2 Examination

Module MS214 Computational Operations Research

Time allowed – 2 hrs Spring Semester 2008

SOLUTIONS

Question 1

A factory produces two kinds of explosives for mining and quarrying, Type I and Type II.

Due to storage problems a maximum of 250 kilograms of explosive can be processed, mixed and packed each week. One kilogram of Type I explosive takes 5 hours processing time and 2 hours mixing and packing time, while Type II explosive takes 2 hours to process and 5 hours to mix and pack. The total processing capacity is 1100 hours per week, the total mixing and packing capacity is 900 hours per week. The profit on one kilogram of Type I explosive is £3 and on one kilogram of Type II explosive is £4.

The Company wants to know how much of each explosive to manufacture in order to maximise their profit per week.

(a) What are the decision variables for this problem? [2]
x_1=number of kilograms of Type I produced per week, x_2=number of kilograms of Type II produced per week

(b) What is the objective function? [2]
The objective function to be maximised is profit per week and is given by $z = 3x_1 + 4x_2$

(c) State the constraints as inequalities. [5]

$$x_1 + x_2 \leq 250 \quad \textit{\textbf{storage capacity}}$$

$$5x_1 + 2x_2 \leq 1100 \quad \textit{\textbf{processing capacity}}$$

$$2x_1 + 5x_2 \leq 900 \quad \textit{\textbf{mixing and packing capacity}}$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

(d) Sketch the feasible region, indicating the coordinates of each of the corner points. On your diagram, show the contour associated with a weekly profit of £1100. [5]
See Fig 1.
The coordinates of the corner points are

$$(0,0), \quad (0,180), \quad (116\frac{2}{3}, 133\frac{1}{3}), \quad (200,50), \quad (220,0)$$

(e) Using this contour or otherwise (but NOT by using the Simplex Method), determine the optimum solution to the problem, justifying your approach. [4]

As the contours of z move away from the origin, z is increasing. The last corner point that z will touch as it leaves the feasible region is $(116\frac{2}{3}, 133\frac{1}{3})$. At this z is at a maximum

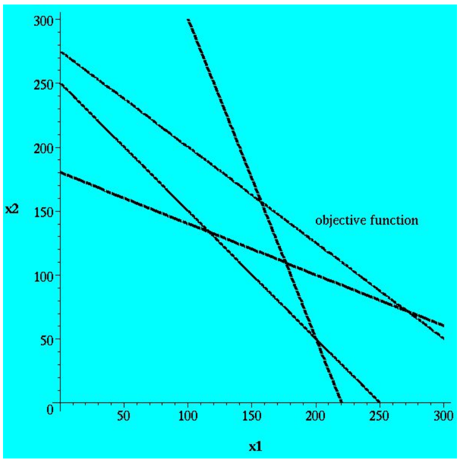

Figure 1: feasible region

alternatively

Calculate the value of the objective function at each corner point of the feasible region. The corner point with the highest value of z will be the optimum solution.

$$z(0,0) = 0, z(0,180) = 720, z(116\frac{2}{3}, 133\frac{1}{3}) = 883\frac{1}{3}, z(200,50) = 800, z(220,0) = 660$$

(f) How much of each type of explosive should the Company manufacture each week? [2]
The company should manufacture $116\frac{2}{3}$ kgs of Type I and $133\frac{1}{3}$ kgs of Type II each week
Note - rounding up or down does not give the optimal solution.

(g) The demand for Type II explosives has risen greatly because of the global mining boom and the Company intends to increase its price. By how much can the profit on Type II explosive be increased before the optimal solution in part (e) changes? [3]
If the profit on Type II is α then $z = 3x_1 + \alpha x_2$. When the gradient of the objective function is less steep than the constraint boundary $2x_1 + 5x_2 = 900$ the optimal corner point will change. The slopes are the same when $\frac{3}{\alpha} = \frac{2}{5}$, i.e. $\alpha = 7\frac{1}{2}$. Profit can be increased by £3.50

(h) If the profit on Type II explosives exceeds this amount, what is your recommendation to the company and why? [2]
The optimal point is now $x_1 = 0, x_2 = 180$ and the profit $z = £1350$. No more of Type II can be made without increasing the packing capacity.

Question 2

(a) Put the following problem in standard form. Do NOT attempt to solve it.

Minimize

$$z = -3x_1 - 4x_2 + 5x_3$$

subject to the constraints

$$\begin{aligned} 4x_1 + 2x_2 - 3x_3 &\geq 3 \\ -2x_1 + 3x_2 + x_3 &\leq -1 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \quad \text{unrestricted}$$

[7]

Turn the problem into a maximisation

$$\textbf{\textit{maximize}} \quad z' = 3x_1 + 4x_2 - 5x_3$$

subject to

$$4x_1 + 2x_2 - 3x_3 - x_4 = 3$$

$$2x_1 - 3x_2 - x_3 - x_5 = 1$$

replace

$$x_3 \quad \textbf{\textit{by}} \quad x_3 = x_3' - x_3''$$

finally substitute for x_3

$$\textbf{\textit{maximise}} \quad z' = 3x_1 + 4x_2 - 5x_3' + 5x_3''$$

$$4x_1 + 2x_2 - 3x_3' + 3x_3'' - x_4 = 3$$

$$2x_1 - 3x_2 - x_3' + x_3'' - x_5 = 1$$

$$x_1, x_2, x_3', x_3'', x_4, x_5 \geq 0$$

(b) Consider the following problem:

Maximize

$$z = 5x_1 + 3x_2$$

subject to the constraints

$$\begin{aligned} 4x_1 + 2x_2 &\leq 8 \\ x_1 + 5x_2 &\leq 6 \end{aligned}$$

and

$$x_1 \geq 0, \quad x_2 \geq 0$$

(i) Write this problem in standard form. [2]

$$\textbf{\textit{Maximize}} \quad z = 5x_1 + 3x_2$$

subject to

$$4x_1 + 2x_2 + x_3 = 8$$

$$x_1 + 5x_2 + x_4 = 6$$

$$x_i \geq 0, \quad i = 1 \ldots 4$$

(ii) Find an initial basic feasible solution, stating which are the basic variables and which are the non-basic variables. [2]
x_1 ***and*** x_2 ***are both zero, they are the non-basic variables.*** $x_3 = 8$ ***and*** $x_4 = 6$ ***are the basic variables.***

(iii) Construct the initial tableau for the Simplex Method. [3]

Basic	z	x_1	x_2	x_3	x_4	Solution
x_3	0	4	2	1	0	8
x_4	0	1	5	0	1	6
z	1	-5	-3	0	0	0

(iv) Use the tableau form of the Simplex Method to find the optimal solution of the problem. At each stage, state the entering and departing variables from the basis and explain your reasoning. [10]

The largest negative value in the z ***row is*** -5 ***so*** x_1 ***is the entering variable. The lowest row quotient is*** $\frac{8}{4} = 2$ ***so*** x_3 ***is the departing variable.***

Basic	z	x_1	x_2	x_3	x_4	Solution
x_1	0	1	$\frac{1}{2}$	$\frac{1}{4}$	0	2
x_4	0	0	$\frac{9}{2}$	$-\frac{1}{4}$	1	4
z	1	0	$-\frac{1}{2}$	$\frac{5}{4}$	0	10

The largest negative value in the z ***row is now*** $-\frac{1}{2}$ ***so*** x_2 ***is the entering variable. The lowest row quotient is*** $\frac{8}{9} = 2$ ***so*** x_4 ***is the departing variable.***

Basic	z	x_1	x_2	x_3	x_4	Solution
x_1	0	1	0	$\frac{5}{18}$	$-\frac{1}{9}$	$\frac{14}{9}$
x_2	0	0	1	$-\frac{1}{18}$	$\frac{2}{9}$	$\frac{8}{9}$
z	1	0	0	$\frac{11}{9}$	$\frac{1}{9}$	$\frac{94}{9}$

(v) From the optimal tableau, determine the optimal value of z and state the values of all the variables at this solution. [1]

The optimal values are $z = \frac{94}{9}, \quad x_1 = \frac{14}{9}, \quad x_2 = \frac{8}{9}, \quad x_3 = 0, \quad x_4 = 0$

Question 3

(a) The optimal tableau of a Linear Programming problem which has two decision variables and two slack variables associated with two constraints is as follows:

Basic	z	x_1	x_2	x_3	x_4	Solution
x_2	0	4	1	0	3	5
x_3	0	5	0	1	2	8
z	1	3	0	0	4	12

An additional constraint is to be added to the problem, given by

$$x_1 - 3x_2 + 3x_3 \leq 4.$$

(i) From the optimal tableau, express the basic variables in terms of the non-basic variables. [2]

$$x_2 = 5 - 4x_1 - 3x_4$$

$$x_3 = 8 - 5x_1 - 2x_4$$

(ii) Express the extra constraint in standard form involving a new slack variable x_5. [1]

$$x_1 - 3x_2 + 3x_3 + x_5 = 4$$

(iii) Rewrite the new constraint in terms of the non-basic variables of the original optimal solution. [2]

$$x_1 - 3(5 - 4x_1 - 3x_4) + 3(8 - 5x_1 - 2x_4) + x_5 = 4$$

which simplifies to

$$-2x_1 + 3x_4 + x_5 = -5$$

(iv) Form a new tableau consisting of the optimal tableau with the extra constraint added. [4]

Basic	z	x_1	x_2	x_3	x_4	x_5	Solution
x_2	0	4	1	0	3	0	5
x_3	0	5	0	1	2	0	8
x_5	0	-2	0	0	3	1	-5
z	1	3	0	0	4	0	12

(v) Determine the entering and departing variables for this new tableau and give your reasoning. [2]
x_5 ***is the departing variable, its solution is negative.*** x_1 ***is the entering variable, its column entry is the most negative.***

(vi) Find the new optimal value of z, assuming that it will be obtained from this first iteration of the Simplex method. [You do NOT have to find the complete optimal tableau.] [2]
We need to add $\frac{3}{2}$ ***times row*** 3 ***to row*** 4***. The*** z ***row becomes***

$$z \left| 1 \right| \quad 0 \quad 0 \quad 0 \quad \frac{17}{2} \quad \frac{3}{2} \left| \frac{9}{2} \right.$$

so the new value of $z = \frac{9}{2}$

(b) The demand for ink at Central Printing Works is 20 litres per day. The print shop manager is able to buy ink for £3 per litre for orders of less than 55 litres and £2.50 per litre for orders of 55 litres or more. Every time an order is placed, a fixed cost of £12 is incurred. The daily cost of storing ink is £0.30 per litre.

Determine the optimal order quantity and the associated cost per day. [12]

β ***is the daily consumption of ink,*** h ***is the daily storage cost,*** c_1 ***and*** $Cc2$ ***the basic and discount prices respectively,*** K ***is the set-up cost and*** q ***is the price break.***

In this case $\beta = 20, \quad h = 0.3, \quad c_1 = 3, \quad c_2 = 2.5, \quad q = 55, \quad K = 12.$ [1]

$$Q_m = \sqrt{\frac{2K\beta}{h}} = \sqrt{\frac{2 \times 12 \times 20}{0.3}} = 40$$

[2]

Thus $q > Q_m$ ***so the optimal quantity is not in Zone 1.*** [1]

We find q_1 ***by solving*** $T_1(Q_m) = T_2(q_1)$ ***where***

$$T_1(Q_m) = \beta c_1 + \frac{K\beta}{Q_m} + \frac{hQ_m}{2}$$

$$T_2(q_1) = \beta c_2 + \frac{K\beta}{q_1} + \frac{hq_1}{2}$$

i.e. [2]

$$20 \times 3 + \frac{12 \times 20}{40} + \frac{0.3 \times 40}{2} = 20 * 2.5 + \frac{12 \times 20}{q_1} + \frac{0.3q_1}{2}$$

$$0.15q_1^2 - 22q_1 + 240 = 0$$

$$q_1 = 11.87 \quad \textbf{\textit{or}} \quad 134.80 \quad \textbf{\textit{to two d.p.s}}$$

[3]

We need the larger value of q_1 ***to find the boundary of Zone 2.*** $q > q_1$ ***so*** $Q_0^\star = q$ [1]

The optimal order quantity is 55 litres. [1]
The total cost is thus $T_1(55) = £62.69$ [1]

Question 4

Three warehouses are supplied by three factories. The supply available from each factory, the demand at each warehouse and the cost per unit of transporting goods from the factories to the warehouses are summarised in the following table:

	Warehouse 1	Warehouse 2	Warehouse 3	Supply
Factory 1	3	6	4	12
Factory 2	5	2	6	16
Factory 3	4	3	3	14
Demand	10	8	24	

(i) What is meant by a *balanced* transportation model? [1]
A balanced transportation model has total supply equal to total demand.

(ii) Briefly describe how a transportation model with excess demand can be made into a balanced model, assuming that there is no penalty for unsatisfied demand. [3]

If demand exceeds supply, a dummy source (i.e. a fictitious factory) is introduced with a capacity equal to the excess demand.
Since the source does not exist, no shipping from that source will occur and so the unit transportation costs can be set to zero.

(iii) Is the above problem balanced or unbalanced? Briefly justify your answer. [2]
The above problem is balanced as the total supply = total demand = 42.

(iv) Use the North-West Corner Method to find an initial basic feasible solution of this problem. [Do NOT use the Least-Cost method] [3]

	W_1	W_2	W_3	***Supply***
F_1	10 3	2 6	 4	***12***
F_2	 5	6 2	10 6	***16***
F_3	 4	 3	14 3	***14***
Demand	***10***	***8***	***24***	

(v) Find the optimal solution of this problem that minimises the transportation costs. [14]

(vi) ***To determine whether the solution is optimal, we calculate the λ_i's, the μ_j's and the s_{ij}'s:***

	3	***6***	***10***
0	10 3	$2-\epsilon$ 6	$+\epsilon$ -6 4
-4	6 5	$6+\epsilon$ 2	$10-\epsilon$ 6
-7	8 4	4 3	14 3

This is not an optimal solution because $s_{13} < 0$***. Thus*** x_{13} ***is the entering variable.*** $\epsilon = 2$
The new tableau with the updated values and the λ_i***'s, the*** μ_j***'s and the*** s_{ij}***'s is:***

	3	***0***	***4***
0	10 3	6 6	2 4
2	0 5	8 2	8 6
-1	2 4	4 3	14 3

There are no negative s_{ij} ***values so this is the optimal tableau.***

(vii) What is the minimum total transportation cost? [2]
The mimimum total transportation cost is $10\times3+2\times4+8\times2+8\times6+14\times3=144$

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