Geometric invariants of the horizontal velocity gradient tensor and their dynamics in shallow water flow

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SUMMARY

Divergence and vorticity are well known to be geometrically invariant quantities in that their mathematical forms are independent of the orientation of the coordinate axes. Various other functions of the elements of the horizontal velocity gradient tensor are invariants in the same sense; examples are the resultant deformation and the determinant and Frobenius norm of the tensor. A brief account of these quadratic invariants is given, focusing on expressions relating them to divergence and vorticity and to one another, on their occurrence in the divergence equation, and on their behaviour under transformation between rotating and non-rotating reference frames. Assuming shallow water dynamics with background rotation, time evolution equations for the resultant deformation and the other quadratic invariants are derived and compared. None rivals the vorticity and potential vorticity equations for compactness, but each may be written quite tidily. Corresponding time evolution equations under quasi-geostrophic shallow water dynamics are also derived, and lead to a simple prognostic equation for the ageostrophic vorticity.

Key Words: kinematics; deformation; divergence equation; potential vorticity; ageostrophic vorticity; Frobenius norm

1. Introduction

A coherent account of the core model equations of contemporary numerical weather prediction and climate simulation need not involve tensor analysis. As numerous textbooks attest, classical definitions of gradient, divergence and curl and a few standard vector differential identities provide the mathematical framework necessary to present the vector momentum equation and the equations of continuity and thermodynamics, and from them to derive the implied conservation equations for angular momentum, energy, vorticity and potential vorticity. See, for example, Gill (1982) and Pedlosky (1987).

This broad-brush picture extends in essence to the primitive equations consistently combined with the shallow atmosphere approximation, although the underlying geometry is then non-Euclidean (Zdunkowski and Bott 2003, pp530, 538; P Müller 2006, p59; Thuburn and White 2013). The various vector differential identities are formally unchanged because the shallow atmosphere approximation is a metric approximation, so derivations and manipulations can continue much as in the Euclidean case (see R Müller 1989; White *et al*. 2005).

Nevertheless, tensor aspects are as important in meteorological dynamics as in general fluid dynamics. At least two broad areas of occurrence may be identified. Realistic representations of viscosity involve the stress tensor, and its divergence features in the momentum equation. Textbooks such as Batchelor (1967) and Acheson (1990) give the appropriate molecular viscous expressions in various curvilinear coordinate systems, while Becker (2001) discusses parallel aspects of the representation of turbulent viscosity (which is the relevant issue in most meteorological modelling). Similar mathematical structures arise when temporal averaging is undertaken in diagnostic studies of the interaction of transient and time-mean fields; see Hoskins *et al*. (1983).

The second tensorial area is fluid kinematics. Here the velocity gradient tensor soon arises because the variation of the flow near any point involves all spatial derivatives of all components of the local flow. Eulerian numerical models routinely disregard kinematics in time integrations (proceeding solely in terms of velocity components and thermodynamic fields – see Salmon 1988) but recourse to kinematics is necessary if explicit trajectories are desired in a forecast or simulation. Also, fluid kinematics features prominently in the semi-Lagrangian time integration method (Staniforth and Côté 1991). This is widely applied in Eulerian models and offers both practical and conceptual advantages in numerical weather prediction and climate simulation (Staniforth *et al*. 2010, Wood *et al*. 2013). Elements of the velocity gradient tensor appear in stability criteria for some semi-Lagrangian schemes (see Pudykiewicz *et al*. 1985).

The velocity gradient tensor occurs also in the time evolution equations for gradients of scalar quantities, and – as will be described in section 3 – some of its geometric invariants feature in the divergence equation as terms that seem obscure if viewed through the usual vectorial prism.

This paper gives an elementary account of the geometric invariants of the velocity gradient tensor (VGT) as it arises in 2D fluid kinematics, and explores their time evolution under shallow water dynamics. Some of these geometric invariants – such as divergence, vorticity and resultant deformation – are well known in meteorology, but others – such as the determinant and Frobenius norm of the VGT – are less familiar. The Frobenius norm of the VGT arises in Thompson's (1980) gravity-wave filtering technique (see section 6, below).

Our study is inspired by the work of Cantwell (1992), Martín *et al*. (1998) and others (see section 4 of the review by Gibbon 2008) on the time evolution of the nine elements of the VGT as it arises in 3D fluid kinematics. In that work, the time dependence is specified by the Euler equations for incompressible flow in the absence of background rotation. Time evolution equations for the geometric invariants of the VGT are constructed from those derived for the individual elements. Typically, solutions of the 3D Euler equations are then obtained, upon assumption of some (fairly restrictive) model of the Hessian matrix of the pressure field. Here we apply essentially the same approach to investigate the time evolution of invariants of the 2D VGT when the divergent shallow water equations with background rotation specify the dynamics. Divergence and rotation are new elements as compared with the 3D studies, and the 2D spatial context makes for an analytically simpler treatment. We do not go on to obtain solutions on the basis of assumptions about the pressure field, but we do examine the time evolution equations that result when quasi-geostrophic dynamics is assumed (and obtain finally a simple prognostic equation for the ageostrophic vorticity).

The paper's structure is as follows. Section 2 notes a familiar decomposition of the 2D VGT and identifies some of the geometric invariants. Section 3 recalls the shallow water equations and the implied vorticity and divergence equations. In sections 4, 5 and 6, the determinant of the VGT, the resultant deformation, and the Frobenius norm of the VGT are discussed and their time evolution equations derived. Section 7 presents quasi-geostrophic, *f*-plane versions of the time evolution equations. Concluding remarks are contained in section 8.

2. Definition, decomposition and geometric invariants of the velocity gradient tensor (VGT)

General accounts of fluid kinematics and the VGT are given in Batchelor (1967), Ottino (1990) and Müller (2006), while treatments in a specifically meteorological context – and more elementary in nature – may be found in Saucier (1955), Wiin-Nielsen (1973) and Bluestein (1992). Zdunkowski and Bott (2003), which uses a dyadic treatment, also has a meteorological context. Assuming 2D flow in the Cartesian *Oxy* plane, we give a brief account of the VGT that covers aspects relevant to later sections.

The elements of the VGT, denoted **A**, are the partial derivatives of the velocity components *u* and *v* with respect to the corresponding coordinates *x* and *y*:

$$\mathbf{A} \equiv \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \tag{1}$$

(The explicit notation $\partial/\partial x$, $\partial/\partial y$ for partial differentiation will also be used here. Note that the 'velocity dyadic' of Zdunkowski and Bott (2003) equates to the transpose of **A** as given by (1).)

The meteorologically familiar divergence, δ, and vorticity, ζ, of the flow are formed from the diagonal and off-diagonal elements of **A**:

$$\delta \equiv u_x + v_y = \mathrm{Trace}(\mathbf{A}) \equiv \mathrm{Tr}(\mathbf{A})\ . \tag{2}$$

$$\zeta \equiv v_x - u_y\ . \tag{3}$$

To first order in distance and time, **A** determines the relative displacements of fluid particles near any point *P* at which its elements are evaluated. In a Cartesian system moving with the flow (*u*, *v*) at *P* and having its origin there at time $t = 0$, the location (x', y') of a particle at time $t = \Delta t$ is (to first order) related to its location (x, y) at time zero by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \Delta t\ . \tag{4}$$

Thus, in terms of the vector locations $\mathbf{x}' \equiv (x', y')$, $\mathbf{x} \equiv (x, y)$ and unit matrix **I**,

$$\mathbf{x}' = (\mathbf{I} + \mathbf{A}\Delta t)\mathbf{x} \tag{5}$$

It is helpful to separate **A** into its antisymmetric and symmetric parts, and also to separate the trace-free component of the symmetric part. Thus, by a simple algebraic decomposition of (1),

$$2\mathbf{A} = \mathbf{R} + \mathbf{S} + \mathbf{D}\,, \tag{6}$$

in which

$$\mathbf{R} = \begin{pmatrix} 0 & -\zeta \\ \zeta & 0 \end{pmatrix} \tag{7}$$

is the antisymmetric part of 2**A**, and $\mathbf{S} + \mathbf{D}$ is its symmetric part:

$$\mathbf{S} = \begin{pmatrix} \delta & 0 \\ 0 & \delta \end{pmatrix}, \tag{8}$$

$$\mathbf{D} = \begin{pmatrix} D_1 & D_2 \\ D_2 & -D_1 \end{pmatrix}. \tag{9}$$

In (9), D_1 and D_2 are the deformation components, given by

$$D_1 \equiv u_x - v_y \;, \quad D_2 \equiv v_x + u_y\,. \tag{10,11}$$

To first order in Δt, (5) can be written as

$$\mathbf{x}' = (\mathbf{I} + ½\mathbf{R}\Delta t)(\mathbf{I} + ½\mathbf{S}\Delta t)(\mathbf{I} + ½\mathbf{D}\Delta t)\mathbf{x} \tag{12}$$

By considering the displacement of a circular ring of fluid particles over the time interval Δt, it can be shown that (as illustrated in Figure 1):
(i) vorticity ζ corresponds to rotation of the ring through an angle $½\zeta\Delta t$;
(ii) divergence δ corresponds to a fractional change $½\delta\Delta t$ of the ring's radius;
(iii) D_1 and D_2 correspond to the deformation of the ring to an ellipse whose major axis is inclined to the *x* axis at an angle $½\tan^{-1}(D_2/D_1)$, its length exceeding the diameter of the initial ring by a fractional amount $½\mathcal{D}\Delta t$ (by which same amount the minor axis falls short of the diameter).
Here $\mathcal{D}$ is the *resultant deformation*, given by

$$\mathcal{D}^2 = D_1^2 + D_2^2 = \left(u_x - v_y\right)^2 + \left(v_x + u_y\right)^2. \tag{13}$$

Some properties of $\mathcal{D}$ are discussed later in this section and in section 5.

It is well known that certain attributes of a (square) matrix are unchanged under similarity transformation (see, for example, Mathews and Walker 1965). In particular, the characteristic equation of the matrix and hence its eigenvalues are unchanged; they are said to be invariants of the matrix, and to be invariant under similarity transformation. For a 2×2 matrix, the trace and determinant are also invariants since they are respectively equal to the sum and the product of the eigenvalues. For $N \times N$ matrices, $N \geq 2$, there are N such invariants.

The tensor **A** has invariance properties under coordinate rotation that subsume those of a 2×2 matrix under similarity transformation. If a coordinate rotation $Oxy \rightarrow OXY$ is made, and the velocity components are also transformed, it is found that the trace of **A**, i.e. the divergence $\delta \equiv u_x + v_y$, is a geometric invariant in that it remains formally unchanged. In outline: if OXY is a Cartesian system rotated anticlockwise with respect to Oxy through an angle α, and velocity components in the OX and OY directions are respectively U and V, then

$$U = u\cos\alpha + v\sin\alpha, \tag{14}$$

$$V = -u\sin\alpha + v\cos\alpha\,. \tag{15}$$

Also,

$$\partial/\partial X = \cos\alpha\,(\partial/\partial x) + \sin\alpha\,(\partial/\partial y), \tag{16}$$

$$\partial/\partial Y = -\sin\alpha\,(\partial/\partial x) + \cos\alpha\,(\partial/\partial y). \tag{17}$$

Use of (15) and (17) to form $\partial U/\partial X$, and of (16) and (18) to form $\partial V/\partial Y$, then shows that

$$\frac{\partial U}{\partial X} + \frac{\partial V}{\partial Y} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\ . \tag{18}$$

In the same way it can readily be shown that the vorticity, ζ, (defined by (3)) is also a geometric invariant (as is expected on physical grounds, given the relationship between vorticity and circulation).

The determinant of **A** is

$$J(u,v) \equiv u_x v_y - v_x u_y\,, \tag{19}$$

i.e. the Jacobian of u, v with respect to x, y. By using just (14)–(17), it can be shown (with some algebraic labour) that $J(u,v)$, as defined by (19), is formally invariant under coordinate rotation, i.e. $J(u,v)$ is a geometric invariant.

$J(u,v)$ arises in the divergence equation (see section 3) but is rarely discussed in meteorology except in a contingent sense. Given its status as an invariant, the scarcity of interest in $J(u,v)$ is surprising. Some properties are noted in section 4, and its time evolution equation under shallow water dynamics is derived.

As is well known, the deformation components D_1 and D_2 (defined by (10) and (11)) are not individually invariant to rotation of the coordinate axes; but $\mathcal{D}$ (see (13)) *is* invariant in this sense. This may be demonstrated by direct transformation (Bluestein 1992, p99), but it also follows from the fact that $\mathcal{D}$ may be expressed in terms of δ, ζ and $J(u,v)$. From the definitions (2), (3), (13) and (19) one obtains

$$\mathcal{D}^2 = \delta^2 + \zeta^2 - 4J(u,v). \tag{20}$$

The invariance of $\mathcal{D}$ then follows simply from that of δ, ζ and $J(u,v)$. Eq.(20) is given by Zdunkowski and Bott (2003), p193.

From a diagnostic viewpoint, the resultant deformation $\mathcal{D}$ and the deformation components D_1 and D_2 are thoroughly treated in textbooks such as Saucier (1955) and Bluestein (1992). However, we know of no discussion of the Lagrangian time evolution of $\mathcal{D}$. This issue is addressed in section 5, once again assuming that the time evolution is governed by the shallow water equations.

Another invariant under coordinate rotation is the sum of the squares of the elements of **A**. This quantity is denoted Q^2 by Thompson (1961, 1980):

$$Q^2 \equiv (u_x)^2 + \left(u_y\right)^2 + (v_x)^2 + \left(v_y\right)^2. \tag{21}$$

Its invariance follows simply from that of $\mathcal{D}$, δ and ζ and from the identity

$$2Q^2 = \mathcal{D}^2 + \delta^2 + \zeta^2 \tag{22}$$

(obtained by applying (2), (3) and (13) in (21)). Use of (20) in (22) leads to an alternative form, given by Thompson (1980) p260:

$$Q^2 = \zeta^2 + \delta^2 - 2J(u,v). \tag{23}$$

From (22) and (23) (or from (13), (19) and (21)) a further form may be obtained:

$$Q^2 = \mathcal{D}^2 + 2J(u,v). \tag{24}$$

This is notable for its simplicity and the absence of the linear invariants δ and ζ.

A physical interpretation of Q^2 may be obtained by forming the squared magnitude σ^2 of the particle displacement over the time interval Δt, according to (5):

$$\sigma^2 \equiv (\mathbf{x}' - \mathbf{x})^{\mathrm{T}}(\mathbf{x}' - \mathbf{x}) = \mathbf{x}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}\mathbf{A}\mathbf{x}\Delta t^2. \tag{25}$$

(The superscript T denotes transposition.) Use in (25) of $\mathbf{x} \equiv (x, y)$ and the definition (1) of **A** leads to

$$\frac{\sigma^2}{\Delta t^2} = \left[(u_x)^2 + \left(u_y\right)^2\right] x^2 + 2\left(u_x u_y + v_x v_y\right) xy + \left[(v_x)^2 + \left(v_y\right)^2\right] y^2. \quad (26)$$

Averaging (26) over the unit circle and use of (21) gives

$$2\,\widehat{\sigma^2} = Q^2 \Delta t^2. \quad (27)$$

Q^2 is thus a measure of the mean squared magnitude of displacements over Δt of particles that initially lie on a circle. The effects of divergence and vorticity, as well as deformation, are taken into account (as is suggested by (22)).

Thompson (1980), p259, called Q^2 'the square of the total deformation'. This form of words does not rest easily with the mean square displacement interpretation just obtained, and might be confused with our use of 'resultant deformation' to denote the quantity $\mathcal{D}$ defined by (13). Since Q^2 is simply the sum of the squares of each element of **A**, we shall use standard matrix vocabulary and call it the Frobenius norm of **A**. [In calling $\mathcal{D}$ the 'resultant deformation' we follow both Saucier (1955) and Bluestein (1992), neither of whom uses the term 'total deformation' – or discusses Q^2.]

The time evolution equation of Q^2 under shallow water dynamics is explored in section 6.

3. Shallow water equations

The *x* and *y* components of the momentum equation for inviscid flow of shallow water of depth *h* above a horizontal bed in a frame rotating at angular rate $f/2$ about the local vertical are

$$\frac{Du}{Dt} - fv = -g\frac{\partial h}{\partial x} \ , \qquad \frac{Dv}{Dt} + fu = -g\frac{\partial h}{\partial y} \ . \tag{28,29}$$

In terms of the divergence δ (see (2)), the accompanying continuity equation is

$$\frac{Dh}{Dt} + h\delta = 0. \tag{30}$$

The material derivative in (28)–(30) is given by

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} \ . \tag{31}$$

From (28) and (29), noting (2), (3) and (31), a prognostic equation for the vorticity ζ follows:

$$\frac{D}{Dt}(\zeta + f) + (\zeta + f)\delta = 0 \ . \tag{32}$$

As is well known and celebrated, (32) condenses to a Lagrangian conservation law upon use of (30):

$$\frac{D}{Dt}\left(\frac{\zeta + f}{h}\right) = 0 \, . \tag{33}$$

The conserved quantity $(\zeta + f)/h$ is the potential vorticity (PV) of the shallow water system.

A prognostic equation for the divergence δ also follows from (28) and (29) (upon use of (2), (3) and (31)). With $\beta \equiv df/dy$ and $\nabla^2 \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ one obtains

$$\frac{D\delta}{Dt} + \delta^2 - 2J(u,v) + \beta u - f\zeta = -g\nabla^2 h \, . \tag{34}$$

Eq.(34) is noticeably more complicated than (32). It condenses slightly upon use of (30) to

$$h\,\frac{D}{Dt}\left(\frac{\delta}{h}\right) - 2J(u,v) + \beta u - f\zeta = -g\nabla^2 h \, . \tag{35}$$

Another useful version of (34) may be obtained by applying identity (23):

$$\frac{D\delta}{Dt} + Q^2 - \zeta^2 + \beta u - f\zeta = -g\nabla^2 h\,. \qquad (36)$$

This form was obtained and applied by Thompson (1980) in a gravity-wave filtering study (as will be discussed further in section 6).

In terms of ageostrophic flow components (u_A, v_A), (28, 29) may be written as

$$\frac{Du}{Dt} - f v_A = 0\ , \qquad \frac{Dv}{Dt} + f u_A = 0\,. \qquad (37, 38)$$

Here

$$u_A \equiv u - u_G\ ,\ \ v_A \equiv v - v_G\ , \qquad (39)$$

where u_G and v_G are the geostrophic flow components defined via

$$f u_G = -g\frac{\partial h}{\partial y}\ \ ,\ \ f v_G = g\frac{\partial h}{\partial x}\ . \qquad (40)$$

The above forms of the divergence equation all simplify slightly when u_A, v_A and the associated vorticity ζ_A are introduced. For example, (35) becomes

$$h\,\frac{D}{Dt}\left(\frac{\delta}{h}\right) - 2J(u,v) + \beta u_A - f\zeta_A = 0\,. \qquad (41)$$

Each form of the divergence equation involves a quadratic invariant of **A**: $J(u,v)$ in (34), (35) and (41); Q^2 in (36). Q^2 may be eliminated from (36) by using (22), but the quadratic invariant $\mathcal{D}^2$ is then present. Without a knowledge of their status as geometric invariants of **A**, these terms in the divergence equation might appear somewhat mysterious.

Finally in this section, we observe how **A** features in the equation for the time evolution of the gradient $\nabla h \equiv \left(h_x\,, h_y\right)^T$. From (30):

$$\frac{D}{Dt}(\nabla h) + \left(u_x h_x + v_x h_y\,, u_y h_x + v_y h_y\right)^T + \nabla(h\delta) = 0. \qquad (42)$$

Use of (1) allows (42) to be written compactly in terms of $\mathbf{A}^T$ as

$$\frac{D}{Dt}(\nabla h) + \mathbf{A}^T\nabla h + \nabla(h\delta) = 0. \qquad (43)$$

4. Determinant: the Jacobian J(u,v)

$J(u,v)$ features prominently in the divergence equation in the forms (34), (35) and (41). Indeed, when and where the flow is precisely geostrophic, $J(u,v) = J(u_G, v_G)$ determines $[(h/2)\times]$ the Lagrangian rate of change of (δ/h), since (41) then reduces to

$$h\frac{D}{Dt}\left(\frac{\delta}{h}\right) = 2J(u_G, v_G). \qquad (44)$$

Cases in which $J(u_G, v_G) = 0$ everywhere include all geostrophically balanced purely zonal flows, but if such symmetry is absent then typically $J(u_G, v_G) \neq 0$.

The term $-2J(u,v)$ obviously remains in (34), (35) and (41) even if the flow is non-divergent ($\delta = 0$), or if $D\delta/Dt$ is omitted (perhaps in order to eliminate gravity waves). In the non-divergent case the velocity field may be represented in terms of a streamfunction, ψ, i.e. $u = -\psi_y$, $v = \psi_x$, and (19) gives

$$J = \psi_{xx}\psi_{yy} - \left(\psi_{xy}\right)^2. \qquad (45)$$

$J(u,v)$ thus reduces in the non-divergent case to the Gaussian curvature of the streamfunction field (multiplied by $\left(1 + (\psi_x)^2 + \left(\psi_y\right)^2\right)^2$). J as given by (45) is familiar as the eponymous nonlinear term in the approximate form of the divergence equation known as the nonlinear balance equation:

$$2\left[\psi_{xx}\psi_{yy} - \left(\psi_{xy}\right)^2\right] + \beta\psi_y + f\nabla^2\psi - g\nabla^2 h = 0. \qquad (46)$$

The corresponding 3D, pressure-coordinate version of (46), is discussed by Daley (1991) and others.

An important aspect of $J(u,v)$ is its behaviour under transformation between co-rotating frames (in particular, rotating and non-rotating frames). Of the quantities ζ, δ, $\mathcal{D}$ and $J(u,v)$ that appear in (20), the vorticity (ζ) is dependent on frame rotation, but the divergence (δ) and the resultant deformation ($\mathcal{D}$) are not. In other words, relative to a non-rotating frame, vorticity has a planetary part but divergence and resultant deformation do not. Consequently, $J(u,v)$ must have a planetary part, and it must be of exactly the form required to accommodate the behaviour of ζ^2 in (20). Similarly, the quantity Q defined by (21) must have a planetary part that exactly accommodates the behaviour of ζ^2 in (22). These indications are confirmed in the Appendix. The situation is fully consistent with the fact that neither (20) nor (22) depends on any assumption about frame rotation rate.

Differentiation of (37, 38) gives prognostic equations for each element of **A**:

$$\frac{D}{Dt}(u_x) + u_x\delta - J(u,v) - f(v_A)_x = 0 \ , \tag{47}$$

$$\frac{D}{Dt}(u_y) + u_y\delta - f(v_A)_y - \beta v_A = 0 \ , \tag{48}$$

$$\frac{D}{Dt}(v_x) + v_x\delta + f(u_A)_x = 0 \ , \tag{49}$$

$$\frac{D}{Dt}(v_y) + v_y\delta - J(u,v) + f(u_A)_y + \beta u_A = 0 \ . \tag{50}$$

From (47)–(50), time evolution equations may be derived for the resultant deformation $\mathcal{D}$ (section 5) and the Frobenius norm Q^2 (section 6) as well as for the Jacobian $J(u,v)$.

Noting the definition (19) of $J(u,v)$, an expression for DJ/Dt can be built up by multiplying (47)–(50) respectively by v_y, $-v_x$, $-u_y$ and u_x, and adding the resulting equations. Some cancellation of terms in $J\delta$ occurs, and upon noting (39) and evident properties of the Jacobian operator, one obtains

$$\frac{DJ}{Dt} + J\delta + f[J(u_G,u_A) + J(v_G,v_A)] + \beta(u_A u_x + v_A v_x) = 0. \tag{51}$$

Use of the continuity equation (30) enables (51) to be written as

$$h\,\frac{D}{Dt}\left[\frac{J(u,v)}{h}\right] + f[J(u_G,u_A) + J(v_G,v_A)] + \beta(u_A u_x + v_A v_x) = 0. \tag{52}$$

Though (52) is more complicated than the potential vorticity equation (33), it does reduce to a conservation law (for $J(u,v)/h$) wherever and whenever the flow is precisely geostrophic. In this respect it is simpler than the divergence equation, which (in the form (41)) reduces to a conservation law (for δ/h) only if $J(u_G,v_G)$ vanishes.

A transparent alternative form of (52) is worth noting for its compactness:

$$h\,\frac{D}{Dt}\left[\frac{J(u,v)}{h}\right] + J(u,fu_A) + J(v,fv_A) = 0. \tag{53}$$

5. Resultant deformation ($\mathcal{D}$)

As noted in section 4 – and shown in the Appendix – the resultant deformation $\mathcal{D}$ is unchanged by transformation between coordinate frames co-rotating at different angular velocities (as indeed are the deformation components D_1 and D_2). There is no 'planetary' part of $\mathcal{D}$. This property is physically reasonable because deformation, as a change of shape, can hardly depend on rotation of the coordinate frame. It is of interest to examine how $\mathcal{D}$ varies following the flow. Is its behaviour simpler than that of δ – which is also independent of frame rotation and yet has a rather complicated Lagrangian rate of change?

An equation for $D\mathcal{D}/Dt$ may be derived by first obtaining expressions for $(D/Dt)D_1$ and $(D/Dt)D_2$ from (47)–(50) and then applying (13) in the differentiated form

$$\mathcal{D}\frac{D\mathcal{D}}{Dt} = D_1\frac{D}{Dt}(D_1) + D_2\frac{D}{Dt}(D_2)\,. \qquad (54)$$

From (47) and (50) (noting (10)):

$$\frac{D}{Dt}(D_1) + D_1\delta - fD_2^A - \beta u_A = 0\,. \qquad (55)$$

From (48) and (49) (noting (11)):

$$\frac{D}{Dt}(D_2) + D_2\delta + fD_1^A - \beta v_A = 0\,. \qquad (56)$$

In (55) and (56),

$$D_1^A \equiv (u_A)_x - (v_A)_y\,, \quad D_2^A \equiv (v_A)_x + (u_A)_y\,, \qquad (57)$$

are the deformation components associated with the ageostrophic flow.

Use of (55) and (56) on the r.h.s. of (54) gives

$$\frac{D}{Dt}\left(\frac{\mathcal{D}^2}{2}\right) + \mathcal{D}^2\delta - f(D_1^G D_2^A - D_2^G D_1^A) - \beta(u_A D_1 + v_A D_2) = 0. \qquad (58)$$

Here

$$D_1^G \equiv (u_G)_x - (v_G)_y\,, \quad D_2^G \equiv (v_G)_x + (u_G)_y\,, \qquad (59)$$

are the deformation components associated with the geostrophic flow.

Upon use of the continuity equation (30), (58) can be re-written as

$$h\,\frac{D}{Dt}\left(\frac{\mathcal{D}^2}{h}\right)+\mathcal{D}^2\delta-2f(D_1^G D_2^A-D_2^G D_1^A)-2\beta(u_A D_1+v_A D_2)=0\,. \quad (60)$$

The most compact form, with no term in δ, has $\mathcal{D}/h$ inside the time derivative:

$$\mathcal{D}h\,\frac{D}{Dt}\left(\frac{\mathcal{D}}{h}\right)-f(D_1^G D_2^A-D_2^G D_1^A)-\beta(u_A D_1+v_A D_2)=0\,. \quad (61)$$

An alternative way of deriving an equation for $D\mathcal{D}/Dt$ is to apply (20) in the differentiated form

$$\mathcal{D}\frac{D\mathcal{D}}{Dt}=\delta\frac{D\delta}{Dt}+\zeta\frac{D\zeta}{Dt}-2\frac{D}{Dt}J(u,v), \quad (62)$$

and then to use expressions already obtained to substitute for the material derivatives appearing on the r.h.s. This affords a much lengthier route to (58), but provides a useful check on the calculations. In the latter connection, the following identity – readily verified – may be noted:

$$D_1^G D_2^A-D_2^G D_1^A=2[J(u_G,u_A)+J(v_G,v_A)]+\zeta_A\delta_G-\zeta_G\delta_A\,. \quad (63)$$

In (63), ζ_A and δ_A are respectively the (relative) vorticity and divergence of the ageostrophic flow, and ζ_G and δ_G similarly for the geostrophic flow. It is clear that (63) applies for any flow vectors (u_G,v_G) and (u_A,v_A) and the associated curls, divergences and deformation components, irrespective of their prior identification with a geostrophic/ageostrophic decomposition.

Identity (63) can be used to write alternative forms of (51) (or (52)) and (58) (or (60), or (61)), though the results are less algebraically compact. Moreover, since it can be shown that $D_1^G D_2^A-D_2^G D_1^A$ is invariant to coordinate rotation, and ζ_A, ζ_G, δ_A, δ_G all enjoy that status, (63) shows that $J(u_G,u_A)+J(v_G,v_A)$ is also invariant to coordinate rotation. This result can be confirmed by direct transformation and repeated use of (14)–(17).

Eq.(61) has a certain symmetry as regards the term involving ageostrophic and geostrophic deformation components. Also, it shares with (52) the property of reducing to a Lagrangian conservation law whenever and wherever the flow is precisely geostrophic. In this respect it too is simpler than the divergence equation (41). [Note that (60), with $\mathcal{D}^2/h$ inside the material derivative, does not exhibit the property in general. However, $\delta_G=0$ if $\beta=0$, so (60) reduces to a Lagrangian conservation law in the geostrophic limit in this case.]

6. Frobenius norm (Q^2)

In a study of gravity-wave filtering, Thompson (1980) sought conditions under which the balanced form of the divergence equation (36) (i.e. (36) with $D\delta/Dt$ set to zero) would be satisfied at all times. To this end he enforced the condition

$$\frac{D}{Dt}(Q^2 - \zeta^2 - f\zeta + g\nabla^2 h) = 0. \qquad (64)$$

[The β-effect was neglected; it had been included in the isentropic coordinate case in Thompson (1961).] Equation (32) delivers $D\zeta/Dt$ for use in (64), and if an equation for DQ^2/Dt can be derived then (64) leads to an equation for Dh/Dt and hence (from (30)) for the divergence δ. Thompson completed this lengthy programme, obtaining in an elegant denouement a weakly nonlinear fourth order elliptic equation for the velocity potential χ ($= \nabla^{-2}\delta$).

Here we derive an equation for DQ^2/Dt directly from (47)–(50) and compare it with Thompson's form. By multiplying (47)–(50) respectively by u_x, u_y, v_x and v_y, adding the results and using (21)–(23) and (40) one finds (when $\beta = 0$),

$$\frac{D}{Dt}Q^2 + (Q^2 + \mathcal{D}^2)\delta - 2f[(\nabla u_G).(\nabla v_A) - (\nabla v_G).(\nabla u_A)] = 0. \qquad (65)$$

In place of the term $(Q^2 + \mathcal{D}^2)\delta$ in (65), Thompson's form (see his p260) has $(3Q^2 - \zeta^2 - \delta^2)\delta$; use of (22) shows that the two terms are equivalent. The explicit scalar product terms in (65) are also equivalent to those obtained by Thompson. Further, if $\beta \neq 0$ then a term $+2\beta\left(u_A v_y - v_A u_y\right)$ appears on the left-hand side of (65), and this term is equivalent to the terms in β retained in Thompson's (1961) isentropic coordinate treatment.

Use of the easily verified identity

$$2[(\nabla u_G).(\nabla v_A) - (\nabla v_G).(\nabla u_A)] = (D_1^G D_2^A - D_2^G D_1^A) + \zeta_A\delta_G - \zeta_G\delta_A \qquad (66)$$

enables (65) to be re-written as

$$\frac{D}{Dt}Q^2 + (Q^2 + \mathcal{D}^2)\delta \; - f[(D_1^G D_2^A - D_2^G D_1^A) + (\zeta_A\delta_G - \zeta_G\delta_A).] = 0. \qquad (67)$$

Identity (66) establishes the invariant status of its left-hand side (since its right-hand side consists of known invariants). In comparison with (65), the form (67) has the advantage of featuring terms already seen (in (58)). Use of (63) in (67) gives another form that involves familiar terms:

$$\frac{D}{Dt}Q^2 + (Q^2 + \mathcal{D}^2)\delta \; - 2f\big[J(u_G, u_A) + J(v_G, v_A) + \zeta_A\delta_G - \zeta_G\delta_A\big] = 0. \quad (68)$$

Equation (67) condenses slightly upon use of the continuity equation (30):

$$h\frac{D}{Dt}\left(\frac{Q^2}{h}\right) + \mathcal{D}^2\delta - f[(D_1^G D_2^A - D_2^G D_1^A) + (\zeta_A\delta_G - \zeta_G\delta_A)] = 0. \quad (69)$$

An alternative form, with Q/h inside the material derivative, is

$$hQ\frac{D}{Dt}\left(\frac{Q}{h}\right) - J\delta - f[(D_1^G D_2^A - D_2^G D_1^A) + (\zeta_A\delta_G - \zeta_G\delta_A)] = 0. \quad (70)$$

Terms in δ are present in both (69) and (70), and in this respect both are rather less tidy than the time evolution equations for J/h (52) and $\mathcal{D}/h$ (61). But since δ_G vanishes when $\beta = 0$ (as has been assumed here) both (69) and (70) reduce to conservation laws in the geostrophic limit in this case.

Identity (24) allows a check on the time-evolution equations (52) for J/h, (60) for $\mathcal{D}^2/h$ and (68) for Q^2/h. Upon using (63) in (68), it can be shown that

$$h\frac{D}{Dt}\left(\frac{Q^2}{h}\right) = h\frac{D}{Dt}\left(\frac{\mathcal{D}^2}{h}\right) + h\frac{D}{Dt}\left(\frac{2J}{h}\right), \quad (71)$$

as required by (24). [The check is easily done if $\beta = 0$. If $\beta \neq 0$, the discarded term $+2\beta(u_A v_y - v_A u_y)$ must be restored to the left-hand side of (68).]

The properties of Q^2 under transformation between different co-rotating frames are established in the Appendix and were discussed in Section 4. Briefly, Q^2 has a planetary part, and it accommodates that of the vorticity so that (22) is obeyed by velocities measured in any member of a set of uniformly co-rotating frames.

7. Quasi-geostrophic (QG) formulation

The treatment of sections 4–6 may be applied using approximate versions of the shallow water equations so long as the component momentum equations are given or deducible. To illustrate procedure and consequences, application using a quasi-geostrophic (QG) shallow water model will be described. For ease of presentation, only the *f*-plane case is considered: the latitude variation of the Coriolis parameter *f* is neglected, so $\beta = 0$ and *f* takes the constant value f_0.

Appropriate *f*-plane QG versions of (37), (38), are

$$\frac{Du_G}{Dt_G} - f_0 v_A = 0\,, \qquad \frac{Dv_G}{Dt_G} + f_0 u_A = 0. \tag{72, 73}$$

The geostrophic flow components u_G and v_G are given by (40) with $f = f_0$, i.e.

$$f_0 u_G = -g\,\partial h/\partial y, \quad f_0 v_G = g\,\partial h/\partial x. \tag{74, 75}$$

The material derivative in (72, 73) is the approximate QG version defined in terms of u_G and v_G as

$$\frac{D}{Dt_G} \equiv \frac{\partial}{\partial t} + u_G \frac{\partial}{\partial x} + v_G \frac{\partial}{\partial y} \quad . \tag{76}$$

The QG form of the continuity equation is

$$\frac{Dh'}{Dt_G} + h_0 \delta_A = 0. \tag{77}$$

Here $h' \equiv h - h_0$, h_0 being an area average value of h. The divergence δ consists only of the contribution δ_A of the ageostrophic flow $u_A \equiv u - u_G$, $v_A \equiv v - v_G$ because the geostrophic flow defined by (74, 75) is non-divergent. For discussion of these QG approximations, see White (2002), p58.

The QG vorticity equation obtained from (72, 73) is

$$\frac{D\zeta_G}{Dt_G} + f_0 \delta_A = 0\ , \tag{78}$$

in which $\zeta_G \equiv \partial v_G/\partial x - \partial u_G/\partial y$ is the vorticity of the geostrophic flow. Upon use of (77) to eliminate δ_A, and (74, 75) to express ζ_G in terms of $h' \equiv h - h_0$, (78) delivers

$$\frac{D}{Dt_G}\left[\left(\nabla^2 - \frac{f_0^2}{gh_0}\right)h'\right] = 0 \qquad (79)$$

The quantity inside the square brackets is the potential vorticity of the QG shallow water system – the 'QGPV' of the system.

Because the geostrophic flow is non-divergent, (72, 73) do not lead to a prognostic equation for the divergence. Gravity waves are not implied. Rather, $\delta = \delta_A$ obeys a simple *diagnostic* partial differential equation, obtained from (77) and (79) by eliminating the local time derivatives:

$$h_0\left(\nabla^2 - \frac{f_0^2}{gh_0}\right)\delta_A = \nabla \cdot [(\mathbf{v}_G \cdot \nabla)\nabla h'] \qquad (80)$$

Taking the divergence of (72, 73) gives another diagnostic relation:

$$-2J(u_G, v_G) - f_0\zeta_A = 0\,. \qquad (81)$$

The vorticity, ζ_A, of the ageostrophic flow, is given explicitly in terms of the geostrophic flow by (81)

As is well known, the QGPV equation (79) determines the time evolution (given suitable boundary conditions). Prognostic equations for QG versions of $J(u, v)$, $\mathcal{D}$ and Q^2 nevertheless follow from (72, 73). The QG analogues of (47)–(50) are

$$\frac{D}{Dt_G}[(u_G)_x] - J(u_G, v_G) - f_0(v_A)_x = 0\ , \qquad (82)$$

$$\frac{D}{Dt_G}[(u_G)_y] - f_0(v_A)_y\ , \quad \frac{D}{Dt_G}[(v_G)_x] + f_0(u_A)_x = 0\,, \qquad (83, 84)$$

$$\frac{D}{Dt_G}[(v_G)_y] - J(u_G, v_G) + f_0(u_A)_y = 0\ . \qquad (85)$$

By manipulating (82)–(85) broadly as in sections 4–6 for the unapproximated shallow water equations, time evolution equations may be derived for the following QG analogues of $J(u, v)$, D_1, D_2, $\mathcal{D}$ and Q^2:

$$J_G \equiv (u_G)_x(v_G)_y - (v_G)_x(u_G)_y\,, \qquad (86)$$

$$D_1^G \equiv (u_G)_x - (v_G)_y\,,\ \ D_2^G \equiv (v_G)_x + (u_G)_y\,, \qquad (87, 88)$$

$$(\mathcal{D}_G)^2 \equiv (D_1^G)^2 + (D_2^G)^2, \tag{89}$$

$$(Q_G)^2 \equiv [(u_G)_x]^2 + \left[(u_G)_y\right]^2 + [(v_G)_x]^2 + \left[(v_G)_y\right]^2. \tag{90}$$

These analogue definitions (86), (87, 88), (89) and (90) correspond to the replacement of u by u_G and v by v_G in (19), (10,11), (13) and (21) respectively. Each analogue is expressible in terms of the streamfunction $\psi_G \equiv gh/f_0$ via $u_G = -(\psi_G)_y$ and $v_G = (\psi_G)_x$. In particular, J_G is expressible in terms of ψ_G in the Gaussian curvature form (45) noted in section 4.

The time evolution equations are:

$$\frac{DJ_G}{Dt_G} + f_0[J(u_G, u_A) + J(v_G, v_A)] = 0 \tag{91}$$

$$\frac{D}{Dt_G}(D_1^G) - f_0 D_2^A = 0, \quad \frac{D}{Dt_G}(D_2^G) + f_0 D_1^A = 0 \tag{92, 93}$$

$$\frac{D}{Dt_G}\left[\frac{(\mathcal{D}_G)^2}{2}\right] - f_0(D_1^G D_2^A - D_2^G D_1^A) = 0 \tag{94}$$

$$\frac{D}{Dt_G}[(Q_G)^2] - 2f_0[J(u_G, u_A) + J(v_G, v_A) - \zeta_G \delta_A] = 0 \tag{95}$$

In (92)–(94), D_1^A and D_2^A are defined as in (56).

Eqs.(91)–(95) may be compared with corresponding forms obtained in sections 4–6. Bearing in mind that $f = f_0$, $\beta = 0$ in the derivation of (91)–(95), there is a close resemblance to (51), (55), (56), (58) and (68) respectively. The absence of terms in δ in (91)–(95) is the main feature of difference; it is a consequence of the non-divergence of the geostrophic flow (u_G, v_G) in the QG model and its appearance as the advecting flow in the QG material derivative (76).

From (92) and (93) (see also (55) and (56)) it may be noted that the ageostrophic deformation components rotate the geostrophic deformation components in a manner reminiscent of inertial oscillations. However, because the ageostrophic deformation components do not necessarily have the same sign as the respective geostrophic components, the analogy is not exact.

An interesting and important feature is that use of (91) after forming D/Dt_G of (81) gives a *prognostic* equation for the ageostrophic vorticity ζ_A :

$$\frac{D\zeta_A}{Dt_G} = 2[J(u_G, u_A) + J(v_G, v_A)]. \quad (96)$$

What would otherwise be an unproductive operation on (81) becomes useful because of (87) and its simplicity – albeit that the Jacobian terms in (96) involve ageostrophic as well as geostrophic flow components, of course. These Jacobian terms may be written in an alternative (if lengthier) form by use of (63).

8. Concluding remarks

This study has examined certain geometric invariants of the 2D velocity gradient tensor. Some of these invariants – such as divergence, vorticity and resultant deformation – are well known meteorological quantities, but others much less so. Examples are the determinant and Frobenius norm of the tensor. The determinant is the Jacobian of the velocity components with respect to the Cartesian coordinates x and y, and the Frobenius norm is the sum of the squares of the four elements of the tensor. Our discussion of these quadratic invariants and of the resultant deformation has: (i) noted their occurrence in the divergence equation (section 3); (ii) identified expressions that relate them to one another (see the compact form (24)) and to the divergence and the vorticity (see (20), (22) and (23)); and (iii) revealed the roles played by the determinant and the Frobenius norm in transformations between coordinate frames rotating with different angular velocities (section 4). Amongst the more obscure geometric invariants to have emerged is the sum of the Jacobian of the x components of two velocity fields and the Jacobian of their y components (both with respect to x, y; see (63)).

Three different sorts of invariance have entered the discussion. Geometric invariance corresponds to covariance under instantaneous rotation of axes. Frame invariance corresponds to covariance under transformation between coordinate frames that rotate with different angular velocities; it amounts to independence of frame rotation rate, and is exhibited by the divergence and the resultant deformation (and, indeed, by its components) but not by the vorticity or by the determinant and the Frobenius norm of the tensor. The third sort of invariance is conservation following the flow – the Lagrangian property exhibited by the potential vorticity (PV) in inviscid flow; see (33).

In sections 4–6, time evolution equations for the quadratic invariants have been derived, the dynamics being supplied by the rotating shallow water equations.

The results provide a useful perspective on the simplicity of the shallow water PV equation (33). This conservation law, as emphasised by Roulstone and Norbury (2013) in a more general context, supplies a constraint that governs the time evolution of flow that is closely geostrophic. In the present study, an assumption of near-geostrophy has been made only in section 7, but throughout it has proved instructive to separate the flow into geostrophic and ageostrophic components. The Jacobian $J(u, v)$ (the tensor's determinant), the total deformation $\mathcal{D}$ and the Frobenius norm Q^2 of the tensor all have time evolution equations that are more complicated than the vorticity equation (32) but nevertheless have reasonably tidy forms. In the absence of a planetary vorticity gradient their behaviour in the geostrophic limit is simpler than that of the divergence equation (34). All five equations may be simplified by use of the continuity equation, and in the case of the vorticity equation (32) the result – as is well known – is the elegant conservation law (33) for potential vorticity.
In section 7 the time evolution equations have been re-derived assuming the dynamics to be governed by *f*-plane, quasi-geostrophic (QG) forms of the shallow water equations (SWEs). A notable feature is the emergence of a simple prognostic equation (96) for the ageostrophic vorticity in terms of the geostrophic and ageostrophic flow components. In other respects also, the results suggest the changes likely when other balanced forms of shallow water dynamics are assumed, such as the semi-geostrophic (SG) model (proposed for baroclinic flow by Hoskins (1975)). This case deserves detailed future study.

In this connection we note that the Jacobian $J(u, v)$ occurs explicitly in *f*-plane SG dynamics as a (numerically) small term supplementing the vorticity of the QG flow:

$$\zeta_{SG} \equiv f_0 + \frac{\partial v_G}{\partial x} - \frac{\partial u_G}{\partial y} + \frac{1}{f_0} J(u_G, v_G). \tag{97}$$

A Monge-Ampère problem arises when a given field of SG potential vorticity ζ_{SG}/h is 'inverted' for the free surface height h, and the Jacobian term in (97) then plays an important role in determining its ellipticity and hence solvability. [As regards both sign and magnitude, the coefficient of the J term in (97) differs from that of similar terms in other formulations, such as the nonlinear balance model (see (46)). McIntyre and Roulstone (2002) discuss this issue in detail.]

It would be interesting to study the consequences of other more comprehensive specifications of the dynamics. Of many candidates, perhaps the most promising cases to consider after shallow water SG are: the baroclinic (3D) QG equations in Cartesian form (e.g. Pedlosky 1987, Vallis 2006); the SWEs in spherical polar form; and the hydrostatic primitive equations (HPEs) in Cartesian and then spherical polar form (with the shallow atmosphere approximation).

Appendix

Transformation of kinematics between rotating and non-rotating frames

In a *non-rotating* Cartesian system *Oxy*, a flow having uniform angular velocity Ω about the origin *O* has velocity components *u* and *v* given by

$$u = -\Omega y, \qquad v = \Omega x\,. \qquad \text{(A1, A2)}$$

From (A1), (A2) and definitions (2), (3), (10), (11), (13), (19) and (21):

$$\delta = 0, \quad \zeta = 2\Omega, \qquad \text{(A3, A4)}$$

$$D_1 = D_2 = \mathcal{D} = 0, \qquad \text{(A5, A6, A7)}$$

$$J = \Omega^2, \quad Q^2 = 2\Omega^2. \qquad \text{(A8, A9)}$$

Only the vorticity (ζ), the Jacobian (J) and the quadratic norm (Q^2) are non-zero. The balance in (20) is between ζ^2 and $4J$, and in (22) between $2Q^2$ and ζ^2.

Consider now a more general flow consisting of a uniform solid rotation, as above, and arbitrary (once differentiable) deviations *U* and *V* from it:

$$u = -\Omega y + U \quad \text{and} \quad v = \Omega x + V. \qquad \text{(A10, A11)}$$

For this flow, application of (2), (3), (10), (11), (13), (19) and (21) gives

$$\delta = U_x + V_y, \quad \zeta = 2\Omega + V_x - U_y\,, \qquad \text{(A12, A13)}$$

$$D_1 = U_x - V_y, \;\; D_2 = V_x + U_y\,, \;\; \mathcal{D}^2 = (D_1)^2 + (D_2)^2, \qquad \text{(A14, A15, A16)}$$

$$J = \Omega^2 + \Omega\left(V_x - U_y\right) + U_x V_y - V_x U_y\;. \qquad \text{(A17)}$$

$$Q^2 = 2\Omega^2 + 2\Omega\left(V_x - U_y\right) + (U_x)^2 + \left(U_y\right)^2 + (V_x)^2 + \left(V_y\right)^2. \qquad \text{(A18)}$$

In (A12), (A14), (A15) and (A16), δ, D_1, D_2 and $\mathcal{D}$ involve only the relative velocity components *U*, *V*. In (A13), ζ contains a ‘planetary’ part 2Ω. In (A.17) and (A.18), terms in Ω^2 and Ω occur; both J and Q^2 contain ‘planetary’ parts. Forming ζ^2 from (A13) shows that its terms in Ω and Ω^2 cancel in (20) with those from $4J$ (see (A17)); thus *relation* (20) *is obeyed by the relative velocity components U and V*. A similar cancellation of terms in Ω and Ω^2 between $2Q^2$ and ζ^2 ensures that relation (22) is obeyed in terms of *U* and *V*.

Applying this argument twice covers transformations between frames rotating at different uniform rates.

The above analysis is 2-dimensional. A 3-dimensional extension would be needed to represent the changes in the direction of apparent vertical that accompany transformations between frames rotating at different angular rates (see White 1982).

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Figure and caption

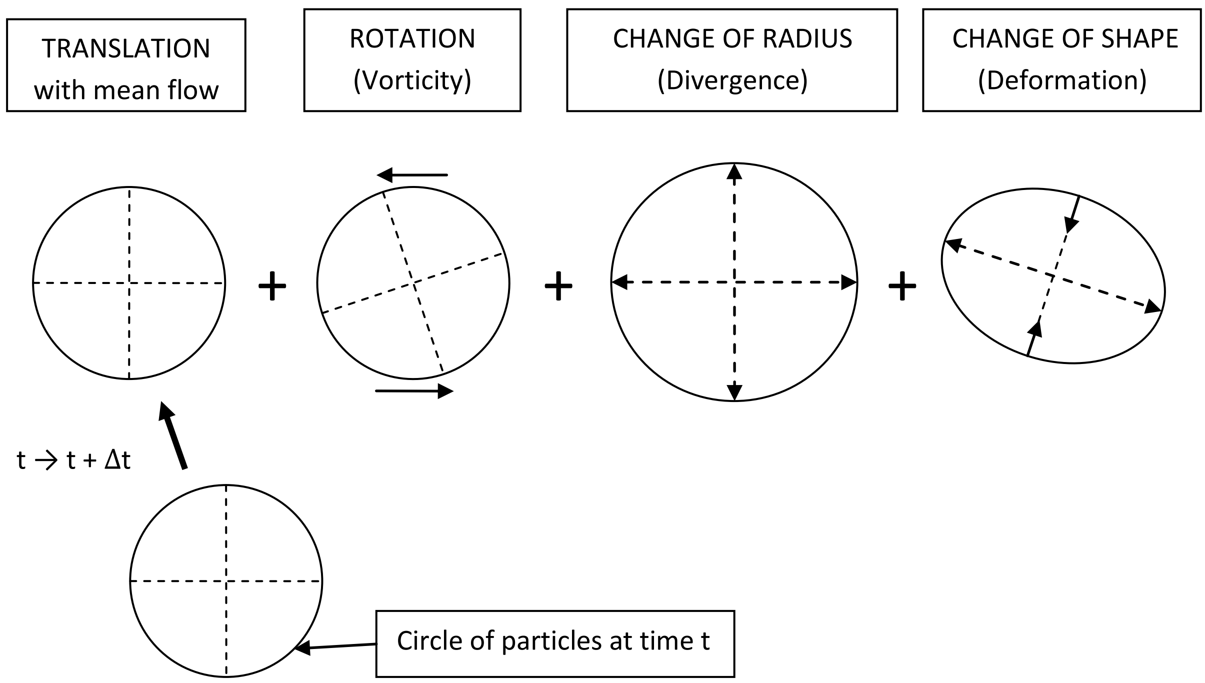

Figure 1 Schematic of a small initial circle of fluid particles undergoing translation, rotation, change of radius and change of shape (circle to ellipse) over time interval Δt. The translation results from the mean flow (the flow at the centre of the circle); the rotation, change of radius and change of shape reflect respectively the vorticity, divergence and deformation of the flow within the circle and its neighbourhood. See White (2002) for further details.