Symplecticity of the Störmer-Verlet algorithm for coupling between shallow-water sloshing and vehicle motion in horizontal plane

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In this technical report it is proved that the Störmer-Verlet algorithm, applied to the dynamically coupled problem of shallow-water sloshing in a vehicle moving in surge-sway directions reported in chapter 8 in [1], is symplectic if the $\sigma_{1,2}$-integrals are discretized using the trapezoidal rule, and considering the following two assumptions

$$p_{1_{i,1}}^{n+\frac{1}{2}} = p_{1_{i,N+1}}^{n+\frac{1}{2}}, \quad \text{for} \quad i=2,\cdots,M\,, \qquad (\textbf{symplectic-1})$$

and

$$p_{2_{1,j}}^{n+\frac{1}{2}} = p_{2_{M+1,j}}^{n+\frac{1}{2}}, \quad \text{for} \quad j=2,\cdots,N\,. \qquad (\textbf{symplectic-2})$$

In continue it is proved that the functional

$$\begin{aligned}
\widehat{\mathscr{H}} \;=\; & \tfrac{1}{2m_v}\left(\tfrac{m_f}{L_1L_2}\left[N\Delta\xi\Delta\eta\dot{q}_1+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}}+p_{1_{i,N+1}}\right)\right.\right.\\
&\left.\left.\qquad +\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{1_{i,j}}\right]-p_3\right)^2\\
&+\tfrac{1}{2m_v}\left(\tfrac{m_f}{L_1L_2}\left[M\Delta\xi\Delta\eta\dot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}}+p_{2_{M+1,j}}\right)\right.\right.\\
&\left.\left.\qquad +\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{2_{i,j}}\right]-p_4\right)^2\\
&+\tfrac{1}{2}\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\dot{q}_1^2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}}^2+p_{1_{i,N+1}}^2\right)+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{1_{i,j}}^2\right)\\
&+\tfrac{1}{2}\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\dot{q}_2^2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}}^2+p_{2_{M+1,j}}^2\right)+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{2_{i,j}}^2\right)\\
&+\tfrac{1}{2}\tfrac{m_f}{L_1L_2}gh_0\sum_{i=1}^{M}\sum_{j=1}^{N}\left(\frac{x_{i+1,j}-x_{i,j}}{\Delta\xi}+\frac{y_{i,j+1}-y_{i,j}}{\Delta\eta}\right)^2\Delta\xi\Delta\eta\\
&+\tfrac{1}{2}\nu_1q_1^2+\tfrac{1}{2}\nu_2q_2^2\,,
\end{aligned}$$

is conserved along the orbits of the semi-discretization (8.7.43) in [1], if

$$\dot{p}_{1_{i,1}}\dot{x}_{i,1} = \dot{p}_{1_{i,N+1}}\dot{x}_{i,N+1}\,, \quad \text{for} \quad i = 2,\cdots,M\,, \tag{\textbf{E-1}}$$

and

$$\dot{p}_{2_{1,j}}\dot{y}_{1,j} = \dot{p}_{2_{M+1,j}}\dot{y}_{M+1,j}\,, \quad \text{for} \quad j = 2,\cdots,N\,. \tag{\textbf{E-2}}$$

1 Checking the symplecticity of the numerical algorithm in §8.8 in [1]

To test for symplecticity, we need the variational equations. The exterior derivative of the discretized equations in (8.8.45) read

$$\begin{aligned}
\delta p_3^{n+\frac{1}{2}} &= \delta p_3^n - \tfrac{\Delta t}{2}\nu_1 \delta q_1^n\,,\\
\delta p_4^{n+\frac{1}{2}} &= \delta p_4^n - \tfrac{\Delta t}{2}\nu_2 \delta q_2^n\,,\\
\delta p_{1_{i,j}}^{n+\frac{1}{2}} &= \delta p_{1_{i,j}}^n + \tfrac{\Delta t}{2} g h_0 \Big(\tfrac{1}{\Delta\xi^2}\left(\delta x_{i+1,j}^n - 2\delta x_{i,j}^n + \delta x_{i-1,j}^n\right)\\
&\quad + \tfrac{1}{\Delta\xi\Delta\eta}\left(\delta y_{i-1,j}^n - \delta y_{i,j+1}^n - \delta y_{i-1,j+1}^n - \delta y_{i,j}^n\right)\Big)\,,\\
&\quad i = 2,\cdots,M, \quad j = 1,\cdots,N\,,\\
\delta p_{2_{i,j}}^{n+\frac{1}{2}} &= \delta p_{2_{i,j}}^n + \tfrac{\Delta t}{2} g h_0 \Big(\tfrac{1}{\Delta\eta^2}\left(\delta y_{i,j+1}^n - 2\delta y_{i,j}^n + \delta y_{i,j-1}^n\right)\\
&\quad + \tfrac{1}{\Delta\xi\Delta\eta}\left(\delta x_{i,j-1}^n - \delta x_{i,j}^n + \delta x_{i+1,j}^n - \delta x_{i+1,j-1}^n\right)\Big)\,,\\
&\quad i = 1,\cdots,M, \quad j = 2,\cdots,N\,, \qquad\qquad (1.1)\\
\delta x_{i,j}^{n+1} &= \delta x_{i,j}^n + \Delta t \left(\delta p_{1_{i,j}}^{n+\frac{1}{2}} - \tfrac{1}{m_v}\delta p_3^{n+\frac{1}{2}} + \tfrac{1}{m_v}\tfrac{m_f}{L_1 L_2}\delta\sigma_1^{n+\frac{1}{2}} \right)\,,\\
&\quad i = 2,\cdots,M, \quad j = 1,\cdots,N\,,\\
\delta y_{i,j}^{n+1} &= \delta y_{i,j}^n + \Delta t \left(\delta p_{2_{i,j}}^{n+\frac{1}{2}} - \tfrac{1}{m_v}\delta p_4^{n+\frac{1}{2}} + \tfrac{1}{m_v}\tfrac{m_f}{L_1 L_2}\delta\sigma_2^{n+\frac{1}{2}} \right)\,,\\
&\quad i = 1,\cdots,M, \quad j = 2,\cdots,N\,,\\
\delta q_1^{n+1} &= \delta q_1^n + \Delta t \left(\tfrac{1}{m_v}\delta p_3^{n+\frac{1}{2}} - \tfrac{1}{m_v}\tfrac{m_f}{L_1 L_2}\delta\sigma_1^{n+\frac{1}{2}} \right)\,,\\
\delta q_2^{n+1} &= \delta q_2^n + \Delta t \left(\tfrac{1}{m_v}\delta p_4^{n+\frac{1}{2}} - \tfrac{1}{m_v}\tfrac{m_f}{L_1 L_2}\delta\sigma_2^{n+\frac{1}{2}} \right)\,,
\end{aligned}$$

$$
\begin{aligned}
\delta p_3^{n+1} &= \delta p_3^{n+\frac{1}{2}} - \tfrac{\Delta t}{2}\nu_1 \delta q_1^{n+1}\,, \\
\delta p_4^{n+1} &= \delta p_4^{n+\frac{1}{2}} - \tfrac{\Delta t}{2}\nu_2 \delta q_2^{n+1}\,, \\
\delta p_{1_{i,j}}^{n+1} &= \delta p_{1_{i,j}}^{n+\frac{1}{2}} + \tfrac{\Delta t}{2} g h_0 \Big(\tfrac{1}{\Delta\xi^2}\left(\delta x_{i+1,j}^{n+1} - 2\delta x_{i,j}^{n+1} + \delta x_{i-1,j}^{n+1}\right) \\
&\quad + \tfrac{1}{\Delta\xi\Delta\eta}\left(\delta y_{i-1,j}^{n+1} - \delta y_{i,j+1}^{n+1} - \delta y_{i-1,j+1}^{n+1} - \delta y_{i,j}^{n+1}\right)\Big)\,, \\
&\quad i = 2, \cdots, M, \quad j = 1, \cdots, N\,, \\
\delta p_{2_{i,j}}^{n+1} &= \delta p_{2_{i,j}}^{n+\frac{1}{2}} + \tfrac{\Delta t}{2} g h_0 \Big(\tfrac{1}{\Delta\eta^2}\left(\delta y_{i,j+1}^{n+1} - 2\delta y_{i,j}^{n+1} + \delta y_{i,j-1}^{n+1}\right) \\
&\quad + \tfrac{1}{\Delta\xi\Delta\eta}\left(\delta x_{i,j-1}^{n+1} - \delta x_{i,j}^{n+1} + \delta x_{i+1,j}^{n+1} - \delta x_{i+1,j-1}^{n+1}\right)\Big)\,, \\
&\quad i = 1, \cdots, M, \quad j = 2, \cdots, N\,.
\end{aligned}
$$

The symplectic form is defined by

$$\mathbf{\Omega}^n = \boldsymbol{\omega}^n + \delta p_3^n \wedge \delta q_1^n + \delta p_4^n \wedge \delta q_2^n\,,$$

with

$$\boldsymbol{\omega}^n = \sum_{i=2}^{M}\sum_{j=1}^{N} a\,\delta p_{1_{i,j}}^n \wedge \delta x_{i,j}^n + \sum_{i=1}^{M}\sum_{j=2}^{N} b\,\delta p_{2_{i,j}}^n \wedge \delta y_{i,j}^n\,,$$

where a and b will be defined shortly. We say that the numerical scheme is symplectic if

$$\mathbf{\Omega}^{n+1} = \mathbf{\Omega}^n \quad \text{for all} \quad n\,.$$

2 The (q_1, p_3) and (q_2, p_4) components of the symplectic form

$$
\begin{aligned}
\delta p_3^{n+1} \wedge \delta q_1^{n+1} &= \delta p_3^{n+\frac{1}{2}} \wedge \delta q_1^{n+1} \\
&= \delta p_3^{n+\frac{1}{2}} \wedge \delta q_1^{n} - e\,\delta p_3^{n+\frac{1}{2}} \wedge \delta \sigma_1^{n+\frac{1}{2}} \\
&= \delta p_3^{n} \wedge \delta q_1^{n} - e\,\delta p_3^{n} \wedge \delta \sigma_1^{n+\frac{1}{2}} + e\tfrac{\Delta t}{2}\nu_1 \delta q_1^n \wedge \delta \sigma_1^{n+\frac{1}{2}}\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\delta p_4^{n+1} \wedge \delta q_2^{n+1} &= \delta p_4^{n+\frac{1}{2}} \wedge \delta q_2^{n+1} \\
&= \delta p_4^{n+\frac{1}{2}} \wedge \delta q_2^{n} - e\,\delta p_4^{n+\frac{1}{2}} \wedge \delta \sigma_2^{n+\frac{1}{2}} \\
&= \delta p_4^{n} \wedge \delta q_2^{n} - e\,\delta p_4^{n} \wedge \delta \sigma_2^{n+\frac{1}{2}} + e\tfrac{\Delta t}{2}\nu_2 \delta q_2^n \wedge \delta \sigma_2^{n+\frac{1}{2}}\,,
\end{aligned}
$$

with $e = \frac{m_f \Delta t}{m_v L_1 L_2}$.

3 The (x, p_1) and (y, p_2) components of the symplectic form

From the definitions; for $i = 2, \cdots, M$ and $j = 1, \cdots, N$,

$$\begin{aligned}
\delta p_{1_{i,j}}^{n+1} \wedge \delta x_{i,j}^{n+1} &= \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta x_{i,j}^{n+1} + c\delta x_{i+1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} + c\delta x_{i-1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} \\
&\quad + d\delta y_{i-1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} + d\delta y_{i,j+1}^{n+1} \wedge \delta x_{i,j}^{n+1} - d\delta y_{i-1,j+1}^{n+1} \wedge \delta x_{i,j}^{n+1} \\
&\quad - d\delta y_{i,j}^{n+1} \wedge \delta x_{i,j}^{n+1} ,
\end{aligned}$$

with $c = \frac{\Delta t g h_0}{2\Delta\xi^2}$ and $d = \frac{\Delta t g h_0}{2\Delta\xi\Delta\eta}$. But

$$\begin{aligned}
\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta x_{i,j}^{n+1} &= \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta x_{i,j}^{n} - \tfrac{\Delta t}{m_v} \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_3^{n+\frac{1}{2}} + e\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_1^{n+\frac{1}{2}} \\
&= \delta p_{1_{i,j}}^{n} \wedge \delta x_{i,j}^{n} + c\delta x_{i+1,j}^{n} \wedge \delta x_{i,j}^{n} + c\delta x_{i-1,j}^{n} \wedge \delta x_{i,j}^{n} \\
&\quad + d\delta y_{i-1,j}^{n} \wedge \delta x_{i,j}^{n} + d\delta y_{i,j+1}^{n} \wedge \delta x_{i,j}^{n} - d\delta y_{i-1,j+1}^{n} \wedge \delta x_{i,j}^{n} \\
&\quad - d\delta y_{i,j}^{n} \wedge \delta x_{i,j}^{n} - \tfrac{\Delta t}{m_v} \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_3^{n} + \tfrac{\Delta t^2}{2m_v} \nu_1 \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_1^{n} \\
&\quad + e\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_1^{n+\frac{1}{2}} ,
\end{aligned}$$

and so

$$\begin{aligned}
\delta p_{1_{i,j}}^{n+1} \wedge \delta x_{i,j}^{n+1} &= \delta p_{1_{i,j}}^{n} \wedge \delta x_{i,j}^{n} + c\delta x_{i+1,j}^{n} \wedge \delta x_{i,j}^{n} + c\delta x_{i-1,j}^{n} \wedge \delta x_{i,j}^{n} \\
&\quad + d\delta y_{i-1,j}^{n} \wedge \delta x_{i,j}^{n} + d\delta y_{i,j+1}^{n} \wedge \delta x_{i,j}^{n} - d\delta y_{i-1,j+1}^{n} \wedge \delta x_{i,j}^{n} \\
&\quad - d\delta y_{i,j}^{n} \wedge \delta x_{i,j}^{n} - \tfrac{\Delta t}{m_v} \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_3^{n} + \tfrac{\Delta t^2}{2m_v} \nu_1 \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_1^{n} \\
&\quad + e\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_1^{n+\frac{1}{2}} + c\delta x_{i+1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} + c\delta x_{i-1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} \\
&\quad + d\delta y_{i-1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} + d\delta y_{i,j+1}^{n+1} \wedge \delta x_{i,j}^{n+1} - d\delta y_{i-1,j+1}^{n+1} \wedge \delta x_{i,j}^{n+1} \\
&\quad - d\delta y_{i,j}^{n+1} \wedge \delta x_{i,j}^{n+1} .
\end{aligned}$$

From the definitions; for $i = 1, \cdots, M$ and $j = 2, \cdots, N$,

$$\begin{aligned}
\delta p_{2_{i,j}}^{n+1} \wedge \delta y_{i,j}^{n+1} &= \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta y_{i,j}^{n+1} + f\delta y_{i,j+1}^{n+1} \wedge \delta y_{i,j}^{n+1} + f\delta y_{i,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1} \\
&\quad + d\delta x_{i,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1} - d\delta x_{i,j}^{n+1} \wedge \delta y_{i,j}^{n+1} + d\delta x_{i+1,j}^{n+1} \wedge \delta y_{i,j}^{n+1} \\
&\quad - d\delta x_{i+1,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1} ,
\end{aligned}$$

with $f = \frac{\Delta t g h_0}{2\Delta\eta^2}$. But

$$\begin{aligned}
\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta y_{i,j}^{n+1} &= \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta y_{i,j}^{n} - \tfrac{\Delta t}{m_v} \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_4^{n+\frac{1}{2}} + e \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta \sigma_2^{n+\frac{1}{2}} \\
&= \delta p_{2_{i,j}}^{n} \wedge \delta y_{i,j}^{n} + f \delta y_{i,j+1}^{n} \wedge \delta y_{i,j}^{n} + f \delta y_{i,j-1}^{n} \wedge \delta y_{i,j}^{n} \\
&\quad + d\delta x_{i,j-1}^{n} \wedge \delta y_{i,j}^{n} - d\delta x_{i,j}^{n} \wedge \delta y_{i,j}^{n} + d\delta x_{i+1,j}^{n} \wedge \delta y_{i,j}^{n} \\
&\quad - d\delta x_{i+1,j-1}^{n} \wedge \delta y_{i,j}^{n} - \tfrac{\Delta t}{m_v} \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_4^{n} \\
&\quad + \tfrac{\Delta t^2}{2m_v} \nu_2 \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_2^{n} + e \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta \sigma_2^{n+\frac{1}{2}},
\end{aligned}$$

and so

$$\begin{aligned}
\delta p_{2_{i,j}}^{n+1} \wedge \delta y_{i,j}^{n+1} &= \delta p_{2_{i,j}}^{n} \wedge \delta y_{i,j}^{n} + f \delta y_{i,j+1}^{n} \wedge \delta y_{i,j}^{n} + f \delta y_{i,j-1}^{n} \wedge \delta y_{i,j}^{n} \\
&\quad + d\delta x_{i,j-1}^{n} \wedge \delta y_{i,j}^{n} - d\delta x_{i,j}^{n} \wedge \delta y_{i,j}^{n} + d\delta x_{i+1,j}^{n} \wedge \delta y_{i,j}^{n} \\
&\quad - d\delta x_{i+1,j-1}^{n} \wedge \delta y_{i,j}^{n} - \tfrac{\Delta t}{m_v} \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_4^{n} \\
&\quad + \tfrac{\Delta t^2}{2m_v} \nu_2 \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_2^{n} + e \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta \sigma_2^{n+\frac{1}{2}} \\
&\quad + f \delta y_{i,j+1}^{n+1} \wedge \delta y_{i,j}^{n+1} + f \delta y_{i,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1} + d\delta x_{i,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1} \\
&\quad - d\delta x_{i,j}^{n+1} \wedge \delta y_{i,j}^{n+1} + d\delta x_{i+1,j}^{n+1} \wedge \delta y_{i,j}^{n+1} - d\delta x_{i+1,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1}.
\end{aligned}$$

4 The symplectic form

Now the symplectic form $\mathbf{\Omega}^{n+1}$ reads

$$\begin{aligned}
\mathbf{\Omega}^{n+1} &= \sum_{i=2}^{M} \sum_{j=1}^{N} a \delta p_{1_{i,j}}^{n+1} \wedge \delta x_{i,j}^{n+1} + \sum_{i=1}^{M} \sum_{j=2}^{N} b \delta p_{2_{i,j}}^{n+1} \wedge \delta y_{i,j}^{n+1} \\
&\quad + \delta p_3^{n+1} \wedge \delta q_1^{n+1} + \delta p_4^{n+1} \wedge \delta q_2^{n+1} \\
&= \boldsymbol{\omega}^n + \sum_{i=2}^{M} \sum_{j=1}^{N} ac \delta x_{i+1,j}^{n} \wedge \delta x_{i,j}^{n} + \sum_{i=2}^{M} \sum_{j=1}^{N} ac \delta x_{i-1,j}^{n} \wedge \delta x_{i,j}^{n} \\
&\quad + \sum_{i=2}^{M} \sum_{j=1}^{N} ad \delta y_{i-1,j}^{n} \wedge \delta x_{i,j}^{n} + \sum_{i=2}^{M} \sum_{j=1}^{N} ad \delta y_{i,j+1}^{n} \wedge \delta x_{i,j}^{n} \\
&\quad - \sum_{i=2}^{M} \sum_{j=1}^{N} ad \delta y_{i-1,j+1}^{n} \wedge \delta x_{i,j}^{n} - \sum_{i=2}^{M} \sum_{j=1}^{N} ad \delta y_{i,j}^{n} \wedge \delta x_{i,j}^{n} \\
&\quad - \sum_{i=2}^{M} \sum_{j=1}^{N} a \frac{\Delta t}{m_v} \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_3^{n} + \sum_{i=2}^{M} \sum_{j=1}^{N} a \frac{\Delta t^2}{2m_v} \nu_1 \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_1^{n}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=2}^{M}\sum_{j=1}^{N} ae\delta {p_1}_{i,j}^{n+\frac{1}{2}} \wedge \delta\sigma_1^{n+\frac{1}{2}} + \sum_{i=2}^{M}\sum_{j=1}^{N} ac\delta x_{i+1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} \\
&+\sum_{i=2}^{M}\sum_{j=1}^{N} ac\delta x_{i-1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} + \sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y_{i-1,j}^{n+1} \wedge \delta x_{i,j}^{n+1} \\
&+\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y_{i,j+1}^{n+1} \wedge \delta x_{i,j}^{n+1} - \sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y_{i-1,j+1}^{n+1} \wedge \delta x_{i,j}^{n+1} \\
&-\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y_{i,j}^{n+1} \wedge \delta x_{i,j}^{n+1} + \sum_{i=1}^{M}\sum_{j=2}^{N} bf\delta y_{i,j+1}^{n} \wedge \delta y_{i,j}^{n} \\
&+\sum_{i=1}^{M}\sum_{j=2}^{N} bf\delta y_{i,j-1}^{n} \wedge \delta y_{i,j}^{n} + \sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x_{i,j-1}^{n} \wedge \delta y_{i,j}^{n} \\
&-\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x_{i,j}^{n} \wedge \delta y_{i,j}^{n} + \sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x_{i+1,j}^{n} \wedge \delta y_{i,j}^{n} \\
&-\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x_{i+1,j-1}^{n} \wedge \delta y_{i,j}^{n} - \sum_{i=1}^{M}\sum_{j=2}^{N} b\frac{\Delta t}{m_v}\delta {p_2}_{i,j}^{n+\frac{1}{2}} \wedge \delta p_4^n \\
&+\sum_{i=1}^{M}\sum_{j=2}^{N} b\frac{\Delta t^2}{2m_v}\nu_2\delta {p_2}_{i,j}^{n+\frac{1}{2}} \wedge \delta q_2^n + \sum_{i=1}^{M}\sum_{j=2}^{N} be\delta {p_2}_{i,j}^{n+\frac{1}{2}} \wedge \delta\sigma_2^{n+\frac{1}{2}} \\
&+\sum_{i=1}^{M}\sum_{j=2}^{N} bf\delta y_{i,j+1}^{n+1} \wedge \delta y_{i,j}^{n+1} + \sum_{i=1}^{M}\sum_{j=2}^{N} bf\delta y_{i,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1} \\
&+\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x_{i,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1} - \sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x_{i,j}^{n+1} \wedge \delta y_{i,j}^{n+1} \\
&+\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x_{i+1,j}^{n+1} \wedge \delta y_{i,j}^{n+1} - \sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x_{i+1,j-1}^{n+1} \wedge \delta y_{i,j}^{n+1} \\
&-e\delta p_3^n \wedge \delta\sigma_1^{n+\frac{1}{2}} + e\tfrac{\Delta t}{2}\nu_1\delta q_1^n \wedge \delta\sigma_1^{n+\frac{1}{2}} - e\delta p_4^n \wedge \delta\sigma_2^{n+\frac{1}{2}} \\
&+e\tfrac{\Delta t}{2}\nu_2\delta q_2^n \wedge \delta\sigma_2^{n+\frac{1}{2}} + \delta p_3^n \wedge \delta q_1^n + \delta p_4^n \wedge \delta q_2^n \,.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{i=2}^{M}\sum_{j=1}^{N} ac\delta x_{i+1,j}^{n} \wedge \delta x_{i,j}^{n} &= \sum_{i=2}^{M-1}\sum_{j=1}^{N} ac\delta x_{i+1,j}^{n} \wedge \delta x_{i,j}^{n} \\
&= \sum_{i=3}^{M}\sum_{j=1}^{N} ac\delta x_{i,j}^{n} \wedge \delta x_{i-1,j}^{n} \,,
\end{aligned}
$$

and

$$\begin{aligned}
\sum_{i=2}^{M}\sum_{j=1}^{N} ac\delta x^n_{i-1,j} \wedge \delta x^n_{i,j} &= \sum_{i=3}^{M}\sum_{j=1}^{N} ac\delta x^n_{i-1,j} \wedge \delta x^n_{i,j} \\
&= -\sum_{i=3}^{M}\sum_{j=1}^{N} ac\delta x^n_{i,j} \wedge \delta x^n_{i-1,j} \,,
\end{aligned}$$

and so

$$\sum_{i=2}^{M}\sum_{j=1}^{N} ac\delta x^n_{i+1,j} \wedge \delta x^n_{i,j} + \sum_{i=2}^{M}\sum_{j=1}^{N} ac\delta x^n_{i-1,j} \wedge \delta x^n_{i,j} = 0 \,.$$

A similar argument proves that

$$\sum_{i=2}^{M}\sum_{j=1}^{N} ac\delta x^{n+1}_{i+1,j} \wedge \delta x^{n+1}_{i,j} + \sum_{i=2}^{M}\sum_{j=1}^{N} ac\delta x^{n+1}_{i-1,j} \wedge \delta x^{n+1}_{i,j} = 0 \,,$$

and

$$\sum_{i=1}^{M}\sum_{j=2}^{N} bf\delta y^n_{i,j+1} \wedge \delta y^n_{i,j} + \sum_{i=1}^{M}\sum_{j=2}^{N} bf\delta y^n_{i,j-1} \wedge \delta y^n_{i,j} = 0 \,,$$

and

$$\sum_{i=1}^{M}\sum_{j=2}^{N} bf\delta y^{n+1}_{i,j+1} \wedge \delta y^{n+1}_{i,j} + \sum_{i=1}^{M}\sum_{j=2}^{N} bf\delta y^{n+1}_{i,j-1} \wedge \delta y^{n+1}_{i,j} = 0 \,.$$

Also note that

$$\begin{aligned}
\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^n_{i-1,j} \wedge \delta x^n_{i,j} &= \sum_{i=2}^{M}\sum_{j=2}^{N} ad\delta y^n_{i-1,j} \wedge \delta x^n_{i,j} \,, \\
\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^n_{i,j+1} \wedge \delta x^n_{i,j} &= \sum_{i=2}^{M}\sum_{j=1}^{N-1} ad\delta y^n_{i,j+1} \wedge \delta x^n_{i,j} \\
&= \sum_{i=2}^{M}\sum_{j=2}^{N} ad\delta y^n_{i,j} \wedge \delta x^n_{i,j-1} \,, \\
-\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^n_{i-1,j+1} \wedge \delta x^n_{i,j} &= -\sum_{i=2}^{M}\sum_{j=1}^{N-1} ad\delta y^n_{i-1,j+1} \wedge \delta x^n_{i,j} \\
&= -\sum_{i=2}^{M}\sum_{j=2}^{N} ad\delta y^n_{i-1,j} \wedge \delta x^n_{i,j-1} \,, \\
-\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^n_{i,j} \wedge \delta x^n_{i,j} &= -\sum_{i=2}^{M}\sum_{j=2}^{N} ad\delta y^n_{i,j} \wedge \delta x^n_{i,j} \,,
\end{aligned}$$

and so the summation gives

$$\begin{aligned}
&\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^n_{i-1,j}\wedge\delta x^n_{i,j}+\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^n_{i,j+1}\wedge\delta x^n_{i,j}\\
&-\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^n_{i-1,j+1}\wedge\delta x^n_{i,j}-\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^n_{i,j}\wedge\delta x^n_{i,j}=\\
&\sum_{i=2}^{M}\sum_{j=2}^{N} ad\left(\delta y^n_{i-1,j}-\delta y^n_{i,j}\right)\wedge\left(\delta x^n_{i,j}-\delta x^n_{i,j-1}\right)\ .
\end{aligned}\tag{4.2}$$

A similar argument proves that

$$\begin{aligned}
&\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^{n+1}_{i-1,j}\wedge\delta x^{n+1}_{i,j}+\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^{n+1}_{i,j+1}\wedge\delta x^{n+1}_{i,j}\\
&-\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^{n+1}_{i-1,j+1}\wedge\delta x^{n+1}_{i,j}-\sum_{i=2}^{M}\sum_{j=1}^{N} ad\delta y^{n+1}_{i,j}\wedge\delta x^{n+1}_{i,j}=\\
&\sum_{i=2}^{M}\sum_{j=2}^{N} ad\left(\delta y^{n+1}_{i-1,j}-\delta y^{n+1}_{i,j}\right)\wedge\left(\delta x^{n+1}_{i,j}-\delta x^{n+1}_{i,j-1}\right)\ ,
\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}
&\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x^n_{i,j-1}\wedge\delta y^n_{i,j}-\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x^n_{i,j}\wedge\delta y^n_{i,j}\\
&+\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x^n_{i+1,j}\wedge\delta y^n_{i,j}-\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x^n_{i+1,j-1}\wedge\delta y^n_{i,j}=\\
&\sum_{i=2}^{M}\sum_{j=2}^{N} bd\left(\delta x^n_{i,j-1}-\delta x^n_{i,j}\right)\wedge\left(\delta y^n_{i,j}-\delta y^n_{i-1,j}\right)\ ,
\end{aligned}\tag{4.4}$$

and

$$\begin{aligned}
&\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x^{n+1}_{i,j-1}\wedge\delta y^{n+1}_{i,j}-\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x^{n+1}_{i,j}\wedge\delta y^{n+1}_{i,j}\\
&+\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x^{n+1}_{i+1,j}\wedge\delta y^{n+1}_{i,j}-\sum_{i=1}^{M}\sum_{j=2}^{N} bd\delta x^{n+1}_{i+1,j-1}\wedge\delta y^{n+1}_{i,j}=\\
&\sum_{i=2}^{M}\sum_{j=2}^{N} bd\left(\delta x^{n+1}_{i,j-1}-\delta x^{n+1}_{i,j}\right)\wedge\left(\delta y^{n+1}_{i,j}-\delta y^{n+1}_{i-1,j}\right)\ .
\end{aligned}\tag{4.5}$$

Now it can be concluded that if $a = b$ then the sum of (4.2), (4.3), (4.4) and (4.5) vanish.

So the symplectic form $\mathbf{\Omega}^{n+1}$ simplifies to

$$\begin{aligned}
\mathbf{\Omega}^{n+1} &= \boldsymbol{\omega}^n - \sum_{i=2}^{M}\sum_{j=1}^{N} a\frac{\Delta t}{m_v}\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_3^n + \sum_{i=2}^{M}\sum_{j=1}^{N} a\frac{\Delta t^2}{2m_v}\nu_1 \delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_1^n \\
&\quad + \sum_{i=2}^{M}\sum_{j=1}^{N} ae\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_1^{n+\frac{1}{2}} - \sum_{i=1}^{M}\sum_{j=2}^{N} a\frac{\Delta t}{m_v}\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_4^n \\
&\quad + \sum_{i=1}^{M}\sum_{j=2}^{N} a\frac{\Delta t^2}{2m_v}\nu_2\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_2^n + \sum_{i=1}^{M}\sum_{j=2}^{N} ae\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_2^{n+\frac{1}{2}} \\
&\quad - e\delta p_3^n \wedge \delta\sigma_1^{n+\frac{1}{2}} + e\tfrac{\Delta t}{2}\nu_1\delta q_1^n \wedge \delta\sigma_1^{n+\frac{1}{2}} - e\delta p_4^n \wedge \delta\sigma_2^{n+\frac{1}{2}} \\
&\quad + e\tfrac{\Delta t}{2}\nu_2\delta q_2^n \wedge \delta\sigma_2^{n+\frac{1}{2}} + \delta p_3^n \wedge \delta q_1^n + \delta p_4^n \wedge \delta q_2^n\,.
\end{aligned}$$

5 Approximating $\sigma_1^{n+\frac{1}{2}}$ and $\sigma_2^{n+\frac{1}{2}}$ with trapezoidal rule

Using the trapezoidal rule to approximate $\sigma_1^{n+\frac{1}{2}}$ and assuming that

$$p_{1_{i,1}}^{n+\frac{1}{2}} = p_{1_{i,N+1}}^{n+\frac{1}{2}}\,, \quad \text{for} \quad i = 2, \cdots, M\,, \qquad \textbf{(symplectic-1)}$$

then from (8.8.48) in [1] it can be concluded that

$$\Delta\xi\Delta\eta \sum_{i=2}^{M}\sum_{j=1}^{N} p_{1_{i,j}}^{n+\frac{1}{2}} = \left(1 + \tfrac{\Delta\xi m_f}{L_1 m_v}\right)\sigma_1^{n+\frac{1}{2}} - \tfrac{\Delta\xi L_2}{m_v} p_3^{n+\frac{1}{2}}\,, \tag{5.6}$$

and using the trapezoidal rule to approximate $\sigma_2^{n+\frac{1}{2}}$ and assuming that

$$p_{2_{1,j}}^{n+\frac{1}{2}} = p_{2_{M+1,j}}^{n+\frac{1}{2}}\,, \quad \text{for} \quad j = 2, \cdots, N\,, \qquad \textbf{(symplectic-2)}$$

then from (8.8.49) in [1] it can be concluded that

$$\Delta\xi\Delta\eta \sum_{i=1}^{M}\sum_{j=2}^{N} p_{2_{i,j}}^{n+\frac{1}{2}} = \left(1 + \tfrac{\Delta\eta m_f}{L_2 m_v}\right)\sigma_2^{n+\frac{1}{2}} - \tfrac{\Delta\eta L_1}{m_v} p_4^{n+\frac{1}{2}}\,. \tag{5.7}$$

Now use (5.6) and (5.7) to simplify the symplectic form $\mathbf{\Omega}^{n+1}$. Note that

$$\begin{aligned}
\sum_{i=2}^{M}\sum_{j=1}^{N} \delta p_{1_{i,j}}^{n+\frac{1}{2}} &= \tfrac{1}{\Delta\xi\Delta\eta}\left(1 + \tfrac{\Delta\xi m_f}{L_1 m_v}\right)\delta\sigma_1^{n+\frac{1}{2}} - \tfrac{L_2}{\Delta\eta m_v}\delta p_3^{n+\frac{1}{2}} \\
&= \tfrac{1}{\Delta\xi\Delta\eta}\left(1 + \tfrac{\Delta\xi m_f}{L_1 m_v}\right)\delta\sigma_1^{n+\frac{1}{2}} - \tfrac{L_2}{\Delta\eta m_v}\delta p_3^n + \tfrac{L_2}{\Delta\eta m_v}\tfrac{\Delta t}{2}\nu_1\delta q_1^n\,,
\end{aligned}$$

and so

$$\sum_{i=2}^{M}\sum_{j=1}^{N} a\frac{\Delta t}{m_v}\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_3^n = \tfrac{a\Delta t}{\Delta\xi\Delta\eta m_v}\left(1+\tfrac{\Delta\xi m_f}{L_1 m_v}\right)\delta\sigma_1^{n+\frac{1}{2}} \wedge \delta p_3^n + \tfrac{a\Delta t^2 L_2}{2\Delta\eta m_v^2}\nu_1\delta q_1^n \wedge \delta p_3^n\,, \tag{5.8}$$

and

$$\sum_{i=2}^{M}\sum_{j=1}^{N} a\frac{\Delta t^2}{2m_v}\nu_1\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_1^n = \tfrac{a\Delta t^2\nu_1}{2m_v\Delta\xi\Delta\eta}\left(1+\tfrac{\Delta\xi m_f}{L_1 m_v}\right)\delta\sigma_1^{n+\frac{1}{2}} \wedge \delta q_1^n - \tfrac{a\Delta t^2\nu_1 L_2}{2\Delta\eta m_v^2}\delta p_3^n \wedge \delta q_1^n\,, \tag{5.9}$$

and

$$\sum_{i=2}^{M}\sum_{j=1}^{N} ae\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_1^{n+\frac{1}{2}} = -\tfrac{aeL_2}{\Delta\eta m_v}\delta p_3^n \wedge \delta\sigma_1^{n+\frac{1}{2}} + \tfrac{aeL_2}{\Delta\eta m_v}\tfrac{\Delta t}{2}\nu_1\delta q_1^n \wedge \delta\sigma_1^{n+\frac{1}{2}}\,. \tag{5.10}$$

A similar argument proves that

$$\sum_{i=1}^{M}\sum_{j=2}^{N} a\frac{\Delta t}{m_v}\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_4^n = \tfrac{a\Delta t}{\Delta\xi\Delta\eta m_v}\left(1+\tfrac{\Delta\eta m_f}{L_2 m_v}\right)\delta\sigma_2^{n+\frac{1}{2}} \wedge \delta p_4^n + \tfrac{a\Delta t^2 L_1}{2\Delta\xi m_v^2}\nu_2\delta q_2^n \wedge \delta p_4^n\,, \tag{5.11}$$

and

$$\sum_{i=1}^{M}\sum_{j=2}^{N} a\frac{\Delta t^2}{2m_v}\nu_2\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_2^n = \tfrac{a\Delta t^2\nu_2}{2m_v\Delta\xi\Delta\eta}\left(1+\tfrac{\Delta\eta m_f}{L_2 m_v}\right)\delta\sigma_2^{n+\frac{1}{2}} \wedge \delta q_2^n - \tfrac{a\Delta t^2\nu_2 L_1}{2\Delta\xi m_v^2}\delta p_4^n \wedge \delta q_2^n\,, \tag{5.12}$$

and

$$\sum_{i=1}^{M}\sum_{j=2}^{N} ae\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_2^{n+\frac{1}{2}} = -\tfrac{aeL_1}{\Delta\xi m_v}\delta p_4^n \wedge \delta\sigma_2^{n+\frac{1}{2}} + \tfrac{aeL_1}{\Delta\xi m_v}\tfrac{\Delta t}{2}\nu_2\delta q_2^n \wedge \delta\sigma_2^{n+\frac{1}{2}}\,. \tag{5.13}$$

Note that

$$\begin{aligned}
&-\sum_{i=2}^{M}\sum_{j=1}^{N} a\frac{\Delta t}{m_v}\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_3^n + \sum_{i=2}^{M}\sum_{j=1}^{N} a\frac{\Delta t^2}{2m_v}\nu_1\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_1^n \\
&+\sum_{i=2}^{M}\sum_{j=1}^{N} ae\delta p_{1_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_1^{n+\frac{1}{2}} - e\delta p_3^n \wedge \delta\sigma_1^{n+\frac{1}{2}} + e\frac{\Delta t}{2}\nu_1\delta q_1^n \wedge \delta\sigma_1^{n+\frac{1}{2}} \\
&= \left(-\tfrac{a\Delta t}{\Delta\xi\Delta\eta m_v}\left(1+\tfrac{\Delta\xi m_f}{L_1 m_v}\right) + \tfrac{aeL_2}{\Delta\eta m_v} + e\right)\delta\sigma_1^{n+\frac{1}{2}} \wedge \delta p_3^n \\
&+ \left(\tfrac{a\Delta t^2\nu_1}{2m_v\Delta\xi\Delta\eta}\left(1+\tfrac{\Delta\xi m_f}{L_1 m_v}\right) - \tfrac{aeL_2}{\Delta\eta m_v}\tfrac{\Delta t}{2}\nu_1 - e\tfrac{\Delta t}{2}\nu_1\right)\delta\sigma_1^{n+\frac{1}{2}} \wedge \delta q_1^n\,,
\end{aligned}$$

now this expression vanishes if

$$a = \frac{m_f \Delta\xi \Delta\eta}{L_1 L_2} = \frac{m_f}{MN} \,.$$

Substituting a into the following expression shows that

$$\begin{aligned}
&-\sum_{i=1}^{M}\sum_{j=2}^{N} a\frac{\Delta t}{m_v}\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta p_4^n + \sum_{i=1}^{M}\sum_{j=2}^{N} a\frac{\Delta t^2}{2m_v}\nu_2 \delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta q_2^n \\
&+\sum_{i=1}^{M}\sum_{j=2}^{N} ae\delta p_{2_{i,j}}^{n+\frac{1}{2}} \wedge \delta\sigma_2^{n+\frac{1}{2}} - e\delta p_4^n \wedge \delta\sigma_2^{n+\frac{1}{2}} + e\frac{\Delta t}{2}\nu_2 \delta q_2^n \wedge \delta\sigma_2^{n+\frac{1}{2}} = 0\,,
\end{aligned}$$

and so the symplectic form $\boldsymbol{\Omega}^{n+1}$ simplifies to

$$\boldsymbol{\Omega}^{n+1} \;=\; \boldsymbol{\omega}^n + \delta p_3^n \wedge \delta q_1^n + \delta p_4^n \wedge \delta q_2^n \,,$$

proving the symplecticity of the scheme with the assumptions (**symplectic-1**) and (**symplectic-2**). In conclusion the symplectic form is

$$\boldsymbol{\Omega}^n = \boldsymbol{\omega}^n + \delta p_3^n \wedge \delta q_1^n + \delta p_4^n \wedge \delta q_2^n \,, \tag{5.14}$$

with

$$\boldsymbol{\omega}^n \;=\; \sum_{i=2}^{M}\sum_{j=1}^{N} \frac{m_f}{MN}\delta p_{1_{i,j}}^n \wedge \delta x_{i,j}^n + \sum_{i=1}^{M}\sum_{j=2}^{N} \frac{m_f}{MN}\delta p_{2_{i,j}}^n \wedge \delta y_{i,j}^n \,.$$

6 Energy of the semi-discretization

Claim: The following functional

$$
\begin{aligned}
\widehat{\mathscr{H}} &= \tfrac{1}{2m_v}\left(\tfrac{m_f}{L_1L_2}\left[N\Delta\xi\Delta\eta\dot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}} + p_{1_{i,N+1}}\right)\right.\right. \\
&\qquad\qquad \left.\left. +\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{1_{i,j}}\right] - p_3\right)^2 \\
&\quad +\tfrac{1}{2m_v}\left(\tfrac{m_f}{L_1L_2}\left[M\Delta\xi\Delta\eta\dot{q}_2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}} + p_{2_{M+1,j}}\right)\right.\right. \\
&\qquad\qquad \left.\left. +\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{2_{i,j}}\right] - p_4\right)^2 \\
&\quad +\tfrac{1}{2}\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\dot{q}_1^2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}}^2 + p_{1_{i,N+1}}^2\right) + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{1_{i,j}}^2\right) \\
&\quad +\tfrac{1}{2}\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\dot{q}_2^2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}}^2 + p_{2_{M+1,j}}^2\right) + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{2_{i,j}}^2\right) \\
&\quad +\tfrac{1}{2}\tfrac{m_f}{L_1L_2}gh_0\sum_{i=1}^{M}\sum_{j=1}^{N}\left(\frac{x_{i+1,j}-x_{i,j}}{\Delta\xi} + \frac{y_{i,j+1}-y_{i,j}}{\Delta\eta}\right)^2\Delta\xi\Delta\eta \\
&\quad +\tfrac{1}{2}\nu_1q_1^2 + \tfrac{1}{2}\nu_2q_2^2\,,
\end{aligned}
\tag{6.15}
$$

is conserved along orbits of the semi-discretization (8.7.43) in [1] if

$$
\dot{p}_{1_{i,1}}\dot{x}_{i,1} = \dot{p}_{1_{i,N+1}}\dot{x}_{i,N+1}\,, \quad \text{for} \quad i = 2,\cdots,M\,, \tag{\textbf{E-1}}
$$

and

$$
\dot{p}_{2_{1,j}}\dot{y}_{1,j} = \dot{p}_{2_{M+1,j}}\dot{y}_{M+1,j}\,, \quad \text{for} \quad j = 2,\cdots,N\,. \tag{\textbf{E-2}}
$$

Proof: Differentiating $\widehat{\mathscr{H}}$,

$$
\begin{aligned}
\frac{\mathrm{d}\widehat{\mathscr{H}}}{\mathrm{d}t} &= \tfrac{1}{m_v}\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\dot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}} + p_{1_{i,N+1}}\right)\right. \\
&\qquad\qquad \left. +\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{1_{i,j}}\right) \\
&\quad \times\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\ddot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}} + \dot{p}_{1_{i,N+1}}\right)\right. \\
&\qquad\qquad \left. +\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{m_v}\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\dot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}}+p_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{2_{i,j}}\right)\\
&\times\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\ddot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}}+\dot{p}_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}\right)\\
&-\tfrac{1}{m_v}\dot{p}_3\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\dot{q}_1+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}}+p_{1_{i,N+1}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{1_{i,j}}\right)\\
&-\tfrac{1}{m_v}p_3\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\ddot{q}_1+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}}+\dot{p}_{1_{i,N+1}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}\right)\\
&-\tfrac{1}{m_v}\dot{p}_4\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\dot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}}+p_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{2_{i,j}}\right)\\
&-\tfrac{1}{m_v}p_4\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\ddot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}}+\dot{p}_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}\right)\\
&+\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\ddot{q}_1\dot{q}_1+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}}p_{1_{i,1}}+\dot{p}_{1_{i,N+1}}p_{1_{i,N+1}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}p_{1_{i,j}}\right)\\
&+\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\ddot{q}_2\dot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}}p_{2_{1,j}}+\dot{p}_{2_{M+1,j}}p_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}p_{2_{i,j}}\right)
\end{aligned}
$$

$$
+\tfrac{m_f}{L_1L_2} g h_0 \sum_{i=1}^{M}\sum_{j=1}^{N}\left[\left(\frac{\dot{x}_{i+1,j}-\dot{x}_{i,j}}{\Delta\xi}+\frac{\dot{y}_{i,j+1}-\dot{y}_{i,j}}{\Delta\eta}\right)\right.
$$

$$
\left.\times\left(\tfrac{x_{i+1,j}-x_{i,j}}{\Delta\xi}+\tfrac{y_{i,j+1}-y_{i,j}}{\Delta\eta}\right)\right]\Delta\xi\Delta\eta
$$

$$
+\tfrac{1}{m_v}\dot{p}_3p_3+\tfrac{1}{m_v}\dot{p}_4p_4+\nu_1\dot{q}_1q_1+\nu_2\dot{q}_2q_2\,.
$$

Noting that

$$
\begin{aligned}
\sum_{i=1}^{M}\sum_{j=1}^{N}\dot{x}_{i+1,j}\left(x_{i+1,j}-x_{i,j}\right)\Delta\xi\Delta\eta &= \sum_{i=2}^{M}\sum_{j=1}^{N}\dot{x}_{i,j}\left(x_{i,j}-x_{i-1,j}\right)\Delta\xi\Delta\eta\,,\\
\sum_{i=1}^{M}\sum_{j=1}^{N}-\dot{x}_{i,j}\left(x_{i+1,j}-x_{i,j}\right)\Delta\xi\Delta\eta &= \sum_{i=2}^{M}\sum_{j=1}^{N}-\dot{x}_{i,j}\left(x_{i+1,j}-x_{i,j}\right)\Delta\xi\Delta\eta\,,\\
\sum_{i=1}^{M}\sum_{j=1}^{N}\dot{x}_{i+1,j}\left(y_{i,j+1}-y_{i,j}\right)\Delta\xi\Delta\eta &= \sum_{i=2}^{M}\sum_{j=1}^{N}\dot{x}_{i,j}\left(y_{i-1,j+1}-y_{i-1,j}\right)\Delta\xi\Delta\eta\,,\\
\sum_{i=1}^{M}\sum_{j=1}^{N}-\dot{x}_{i,j}\left(y_{i,j+1}-y_{i,j}\right)\Delta\xi\Delta\eta &= \sum_{i=2}^{M}\sum_{j=1}^{N}-\dot{x}_{i,j}\left(y_{i,j+1}-y_{i,j}\right)\Delta\xi\Delta\eta\,,\\
\sum_{i=1}^{M}\sum_{j=1}^{N}\dot{y}_{i,j+1}\left(x_{i+1,j}-x_{i,j}\right)\Delta\xi\Delta\eta &= \sum_{i=1}^{M}\sum_{j=2}^{N}\dot{y}_{i,j}\left(x_{i+1,j-1}-x_{i,j-1}\right)\Delta\xi\Delta\eta\,,\\
\sum_{i=1}^{M}\sum_{j=1}^{N}-\dot{y}_{i,j}\left(x_{i+1,j}-x_{i,j}\right)\Delta\xi\Delta\eta &= \sum_{i=1}^{M}\sum_{j=2}^{N}-\dot{y}_{i,j}\left(x_{i+1,j}-x_{i,j}\right)\Delta\xi\Delta\eta\,,\\
\sum_{i=1}^{M}\sum_{j=1}^{N}\dot{y}_{i,j+1}\left(y_{i,j+1}-y_{i,j}\right)\Delta\xi\Delta\eta &= \sum_{i=1}^{M}\sum_{j=2}^{N}\dot{y}_{i,j}\left(y_{i,j}-y_{i,j-1}\right)\Delta\xi\Delta\eta\,,\\
\sum_{i=1}^{M}\sum_{j=1}^{N}-\dot{y}_{i,j}\left(y_{i,j+1}-y_{i,j}\right)\Delta\xi\Delta\eta &= \sum_{i=1}^{M}\sum_{j=2}^{N}-\dot{y}_{i,j}\left(y_{i,j+1}-y_{i,j}\right)\Delta\xi\Delta\eta\,,
\end{aligned}
$$

and that

$$
\begin{aligned}
\tfrac{1}{m_v}\dot{p}_3p_3 &= \dot{p}_3\dot{q}_1+\tfrac{1}{m_v}\dot{p}_3\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\dot{q}_1+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}}+p_{1_{i,N+1}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{1_{i,j}}\right)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\tfrac{1}{m_v}\dot{p}_4p_4 &= \dot{p}_4\dot{q}_2+\tfrac{1}{m_v}\dot{p}_4\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\dot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}}+p_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{2_{i,j}}\right)\,,
\end{aligned}
$$

and using the sixth and eighth of (8.7.43) in [1], then $\frac{\mathrm{d}\widehat{\mathscr{H}}}{\mathrm{d}t}$ simplifies to

$$
\begin{aligned}
\frac{\mathrm{d}\widehat{\mathscr{H}}}{\mathrm{d}t} = & \frac{1}{m_v}\frac{m_f}{L_1 L_2}\Bigg(N\Delta\xi\Delta\eta\dot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}} + p_{1_{i,N+1}}\right) \\
& \qquad + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N} p_{1_{i,j}}\Bigg) \\
& \times \frac{m_f}{L_1 L_2}\Bigg(N\Delta\xi\Delta\eta\ddot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}} + \dot{p}_{1_{i,N+1}}\right) \\
& \qquad + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N} \dot{p}_{1_{i,j}}\Bigg) \\
& + \frac{1}{m_v}\frac{m_f}{L_1 L_2}\Bigg(M\Delta\xi\Delta\eta\dot{q}_2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}} + p_{2_{M+1,j}}\right) \\
& \qquad + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N} p_{2_{i,j}}\Bigg) \\
& \times \frac{m_f}{L_1 L_2}\Bigg(M\Delta\xi\Delta\eta\ddot{q}_2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}} + \dot{p}_{2_{M+1,j}}\right) \\
& \qquad + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N} \dot{p}_{2_{i,j}}\Bigg) \\
& - \frac{1}{m_v}p_3\frac{m_f}{L_1 L_2}\Bigg(N\Delta\xi\Delta\eta\ddot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}} + \dot{p}_{1_{i,N+1}}\right) \\
& \qquad + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N} \dot{p}_{1_{i,j}}\Bigg) \\
& - \frac{1}{m_v}p_4\frac{m_f}{L_1 L_2}\Bigg(M\Delta\xi\Delta\eta\ddot{q}_2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}} + \dot{p}_{2_{M+1,j}}\right) \\
& \qquad + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N} \dot{p}_{2_{i,j}}\Bigg) \\
& + \frac{m_f}{L_1 L_2}\Bigg(N\Delta\xi\Delta\eta\ddot{q}_1\dot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}}p_{1_{i,1}} + \dot{p}_{1_{i,N+1}}p_{1_{i,N+1}}\right) \\
& \qquad + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N} \dot{p}_{1_{i,j}}p_{1_{i,j}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\ddot{q}_2\dot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}}p_{2_{1,j}}+\dot{p}_{2_{M+1,j}}p_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}p_{2_{i,j}}\right)\\
&+\tfrac{m_f}{L_1L_2}gh_0\sum_{i=2}^{M}\sum_{j=1}^{N}\dot{x}_{i,j}\left(\frac{-x_{i+1,j}+2x_{i,j}-x_{i-1,j}}{\Delta\xi^2}\right.\\
&\qquad\left.+\tfrac{y_{i-1,j+1}-y_{i-1,j}-y_{i,j+1}+y_{i,j}}{\Delta\xi\Delta\eta}\right)\Delta\xi\Delta\eta\\
&+\tfrac{m_f}{L_1L_2}gh_0\sum_{i=1}^{M}\sum_{j=2}^{N}\dot{y}_{i,j}\left(\frac{-y_{i,j+1}+2y_{i,j}-y_{i,j-1}}{\Delta\eta^2}\right.\\
&\qquad\left.+\tfrac{x_{i+1,j-1}-x_{i,j-1}-x_{i+1,j}+x_{i,j}}{\Delta\xi\Delta\eta}\right)\Delta\xi\Delta\eta\,.
\end{aligned}
$$

Noting that

$$
\begin{aligned}
&\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\ddot{q}_1\dot{q}_1+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}}p_{1_{i,1}}+\dot{p}_{1_{i,N+1}}p_{1_{i,N+1}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}p_{1_{i,j}}\right)\\
&=\tfrac{m_f}{L_1L_2}\left(\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}}\dot{x}_{i,1}+\dot{p}_{1_{i,N+1}}\dot{x}_{i,N+1}\right)+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}\dot{x}_{i,j}\right)\\
&+\tfrac{m_f}{L_1L_2}\dot{q}_1\left(N\Delta\xi\Delta\eta\ddot{q}_1+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}}+\dot{p}_{1_{i,N+1}}\right)+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}\right)\,,
\end{aligned}
$$

and that

$$
\begin{aligned}
&\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\ddot{q}_2\dot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}}p_{2_{1,j}}+\dot{p}_{2_{M+1,j}}p_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}p_{2_{i,j}}\right)\\
&=\tfrac{m_f}{L_1L_2}\left(\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}}\dot{y}_{1,j}+\dot{p}_{2_{M+1,j}}\dot{y}_{M+1,j}\right)+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}\dot{y}_{i,j}\right)\\
&+\tfrac{m_f}{L_1L_2}\dot{q}_2\left(M\Delta\xi\Delta\eta\ddot{q}_2+\tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}}+\dot{p}_{2_{M+1,j}}\right)+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}\right)\,,
\end{aligned}
$$

now if

$$
\boxed{\dot{p}_{1_{i,1}}\dot{x}_{i,1}=\dot{p}_{1_{i,N+1}}\dot{x}_{i,N+1}\,,\quad\text{for}\quad i=2,\cdots,M\,,}
$$

then from the second equation in (8.7.43) in [1] it can be concluded that

$$\frac{m_f}{L_1L_2}\left(\frac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}}\dot{x}_{i,1}+\dot{p}_{1_{i,N+1}}\dot{x}_{i,N+1}\right)+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}\dot{x}_{i,j}\right)$$
$$+\frac{m_f}{L_1L_2}gh_0\sum_{i=2}^{M}\sum_{j=1}^{N}\dot{x}_{i,j}\left(\frac{-x_{i+1,j}+2x_{i,j}-x_{i-1,j}}{\Delta\xi^2}\right.$$
$$\left.+\frac{y_{i-1,j+1}-y_{i-1,j}-y_{i,j+1}+y_{i,j}}{\Delta\xi\Delta\eta}\right)\Delta\xi\Delta\eta=0\,,$$

and if

$$\boxed{\dot{p}_{2_{1,j}}\dot{y}_{1,j}=\dot{p}_{2_{M+1,j}}\dot{y}_{M+1,j}\,,\quad\text{for}\quad j=2,\cdots,N\,,}$$

then from the fourth equation in (8.7.43) in [1] it can be concluded that

$$\frac{m_f}{L_1L_2}\left(\frac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}}\dot{y}_{1,j}+\dot{p}_{2_{M+1,j}}\dot{y}_{M+1,j}\right)+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}\dot{y}_{i,j}\right)$$
$$+\frac{m_f}{L_1L_2}gh_0\sum_{i=1}^{M}\sum_{j=2}^{N}\dot{y}_{i,j}\left(\frac{-y_{i,j+1}+2y_{i,j}-y_{i,j-1}}{\Delta\eta^2}\right.$$
$$\left.+\frac{x_{i+1,j-1}-x_{i,j-1}-x_{i+1,j}+x_{i,j}}{\Delta\xi\Delta\eta}\right)\Delta\xi\Delta\eta=0\,.$$

Hence $\frac{\mathrm{d}\widehat{\mathscr{H}}}{\mathrm{d}t}$ simplifies to

$$\begin{aligned}\frac{\mathrm{d}\widehat{\mathscr{H}}}{\mathrm{d}t} &= \frac{1}{m_v}\frac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\dot{q}_1+\frac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(p_{1_{i,1}}+p_{1_{i,N+1}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{1_{i,j}}\right)\\
&\quad\times\frac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\ddot{q}_1+\frac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}}+\dot{p}_{1_{i,N+1}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}\right)\\
&\quad+\frac{1}{m_v}\frac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\dot{q}_2+\frac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(p_{2_{1,j}}+p_{2_{M+1,j}}\right)\right.\\
&\qquad\left.+\Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}p_{2_{i,j}}\right)\end{aligned}$$

$$
\begin{aligned}
&\times \tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\ddot{q}_2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}} + \dot{p}_{2_{M+1,j}}\right)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left. + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}\right)\\
&-\tfrac{1}{m_v}p_3\tfrac{m_f}{L_1L_2}\left(N\Delta\xi\Delta\eta\ddot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}} + \dot{p}_{1_{i,N+1}}\right)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left. + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}\right)\\
&-\tfrac{1}{m_v}p_4\tfrac{m_f}{L_1L_2}\left(M\Delta\xi\Delta\eta\ddot{q}_2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}} + \dot{p}_{2_{M+1,j}}\right)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left. + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}\right)\\
&+\tfrac{m_f}{L_1L_2}\dot{q}_1\left(N\Delta\xi\Delta\eta\ddot{q}_1 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{i=2}^{M}\left(\dot{p}_{1_{i,1}} + \dot{p}_{1_{i,N+1}}\right)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left. + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{1_{i,j}}\right)\\
&+\tfrac{m_f}{L_1L_2}\dot{q}_2\left(M\Delta\xi\Delta\eta\ddot{q}_2 + \tfrac{1}{2}\Delta\xi\Delta\eta\sum_{j=2}^{N}\left(\dot{p}_{2_{1,j}} + \dot{p}_{2_{M+1,j}}\right)\right.\\
&\qquad\qquad\qquad\qquad\qquad\qquad\left. + \Delta\xi\Delta\eta\sum_{i=2}^{M}\sum_{j=2}^{N}\dot{p}_{2_{i,j}}\right),
\end{aligned}
$$

and this expression vanishes using the fifth and seventh equations in (8.7.43) in [1].

References

[1] H. Alemi Ardakani. *Rigid-body motion with interior shallow-water sloshing*, PhD Thesis in Mathematics, University of Surrey, UK (2010).