Reduced Dynamics for Momentum Maps with Cocycles

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Abstract We generalize the results on the explicit form of Hamilton's equations in local coordinates near a group orbit of Roberts, Wulff and Lamb to local momentum maps with cocycles. We use these equations to study the dynamics near relative equilibria.

Dynamique réduite pour les applications moment avec des cocycles

Résumé - On généralise les résultats concernants la forme explicite des équations hamiltoniennes dans des coordonnées locales près d'une orbite de groupe de Roberts, Wulff et Lamb pour les applications moment locales avec des cocycles. On utilise ces équations pour étudier la dynamique près d'équilibres relatifs.

Version française abrégée - On considère une variété symplectique de dimension finie $(\mathcal{M},\omega)$ et une fonction de Hamilton $H:\mathcal{M}\to\mathbb{R}$. Soit G un groupe de Lie de dimension finie qui agit proprement et de façon symplectique sur $\mathcal{M}$ et qui laisse la fonction H invariante. Par le théorème de Noether, la partie continue du groupe de symétrie force le système hamiltonien à posseder des quantités conservées, appelées moments. Pour tout $p\in\mathcal{M}$, il existe un voisinage ouvert $\mathcal{U}_p$ contenant p tel que le champ de vecteurs $\xi\mapsto\xi_{\mathcal{M}}(p)$ est hamiltonien pour tout $\xi\in\mathbf{g}=T_{id}G$ avec fonction de Hamilton $\mathbf{J}_\xi:\mathcal{U}_p\to\mathbb{R}$. La fonction $\mathbf{J}_\xi$ peut être choisie linéaire en ξ et définit donc une *application moment* $\mathbf{J}:\mathcal{U}_p\to\mathbf{g}^*$ par $\langle\mathbf{J}(p),\xi\rangle=\mathbf{J}_\xi(p)$ où $\langle,\rangle$ désigne la forme bilinéaire canonique sur $\mathbf{g}^*\times\mathbf{g}$.
Dans cette note on s'intéresse aux applications moment qui ne peuvent pas être choisies équivariantes (sur le plan infinitésimal) par rapport à l'action co-adjointe et on souligne les différences causées par des cocycles non-trivials de l'application moment. On définit alors le *cocycle infinitésimal* $\Sigma:\mathbf{g}\times\mathbf{g}\to\mathbb{R}$ par $\Sigma(\xi,\eta)=\mathbf{J}_{[\xi,\eta]}(x)-\{\mathbf{J}_\xi,\mathbf{J}_\eta\}(x)$ pour $\xi,\eta\in\mathbf{g}$ et $x\in\mathcal{U}_p$ arbitraire (voir [4]). Il s'en suit que l'application moment est équivariante sur le plan infinitésimal par rapport à l'action suivante dépendante de Σ de $\mathbf{g}$ sur $\mathbf{g}^*$:

$$\xi_{\mathbf{g}^*,\Sigma}(\mu)=-ad_\xi^*\mu+\Sigma(\xi,\cdot)\quad\text{pour}\quad\xi\in\mathbf{g}.$$

Ces définitions seront utilisées pour généraliser les résultats sur la forme explicite des équations hamiltoniennes dans des coordonnées locales près d'une orbite de groupe (voir Roberts, Wulff et Lamb [7]) pour les applications moment locales avec des cocycles. Ces coordonnées locales servent spécialement à étudier la stabilité et les bifurcations d'équilibres relatifs.
Suivant une construction analogue à celle dans [7], on commence par regarder un modèle local de $\mathcal{M}$ pour en déduire ensuite la forme des équations hamiltoniennes près d'une orbite de groupe. Par le théorème de section de Palais [6], un voisinage $U\subset\mathcal{M}$ invariante sous l'action de G d'une orbite Gp est difféomorphe à un voisinage de $p\cong(id,0)$ dans $G\times_{G_p}N$ invariante sous l'action de G où N désigne l'espace normal de $T_p\mathcal{M}$ et on identifie $(gg_p^{-1},g_pw)=(g,w)$ pour $g\in G$, $g_p\in G_p$, $w\in N$. On note par N_1 le sous-espace symplectique maximal de N et par $\mathbf{g}_\mu^\Sigma=\{\xi\in\mathbf{g}\mid\xi_{\mathbf{g}^*,\Sigma}\,\mu=0\}$ l'algèbre d'isotropie de μ. Alors pour $\mu=\mathbf{J}(p)$ la variété $\tilde{\mathcal{M}}_0=G\times((\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^*\oplus N_1)\cong G\times N$ constitue un modèle local de $\mathcal{M}$ au voisinage d'une orbite Gp. Regardons la variété $\tilde{\mathcal{M}}=G\times((\mathbf{g}_\mu^\Sigma)^*\oplus N_1)$ munie d'une 2-forme $\tilde\omega$ qui est isomorphe sur $T_{(id,0,0)}\tilde{\mathcal{M}}_0$ à $\omega(p)$. Grace au théorème de Darboux relatif, il existe même un symplectomorphisme entre un voisinage de $G\times_{G_p}\{0\}$ dans $G\times_{G_p}N$ et un voisinage $U\subset\mathcal{M}$ de Gp.

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La fonction de Hamilton H regardée dans les coordonnées locales au voisinage de $G \times_{G_p} \{0\}$ dans $G \times_{G_p} N$ est donc une fonction h qui ne dépend que des variables dans l'espace normale $(\nu, w) \in (\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^* \oplus N_1 \cong N$. On définit une extension de h sur $(\mathbf{g}_\mu^\Sigma)^* \oplus N_1$ par $\tilde{h}(\zeta, w) = h(P_{(\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^*}\zeta, w)$ où $P_{(\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^*} = id - P_{\mathbf{g}_p^*}$ et $P_{\mathbf{g}_p^*}$ désigne la projection sur l'annihilateur d'un complément $\mathbf{m}_\mu^\Sigma$ invariant sous G_p de $\mathbf{g}_p$ dans $\mathbf{g}$: $\mathbf{g}_p^* \cong \mathrm{ann}_{(\mathbf{g}_\mu^\Sigma)^*}(\mathbf{m}_\mu^\Sigma)$. Notant par $\mathbf{J}_{N_1} : N_1 \to \mathbf{g}_p^*$ l'application moment pour l'action symplectique de G_p sur N_1, on obtient le théorème suivant:

Théorème 1 *Dans les coordonnées locales (g, ν, w) de la paramétrisation du théorème de section de Palais et avec $\zeta = \nu + \mathbf{J}_{N_1}(w)$, les équations hamiltoniennes dans un voisinage de Gp invariant sous l'action de G sont données par (11)–(13) où l'application $i_\mu : (\mathbf{g}_\mu^\Sigma)^* \to \mathbf{g}^*$ est induite par le complément $\mathbf{n}_\mu^\Sigma$ de $\mathbf{g}_\mu^\Sigma$ dans $\mathbf{g}$ et $j_\mu(\zeta) : \mathbf{g}_\mu^\Sigma \to \mathbf{g}$ est définie par $j_\mu(\zeta)\xi = \xi + \eta_\mu(\xi, \zeta)$ avec $\eta = \eta_\mu(\xi, \zeta) \in \mathbf{n}_\mu^\Sigma$ étant la solution unique de l'équation (10).*

L'équation pour ν se déduit de ce théorème en projetant l'équation pour ζ sur $(\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^*$.

Si l'application moment est définie sur un voisinage $\mathcal{U}$ de Gp invariant sous l'action de G, alors elle est équivariante par rapport à l'action $g \cdot_\sigma \mu := Ad^*_{g^{-1}}\mu + \sigma(g)$. Le *cocycle* $\sigma : G \to \mathbf{g}^*$ est défini par $\sigma(g) = \mathbf{J}(gp) - Ad^*_{g^{-1}}\mathbf{J}(p)$ pour $g \in G$ et $p \in \mathcal{U}$ arbitraire, et $\Sigma = D\sigma_\eta(id)\xi$ (voir [2, 4]). Dans les coordonnées locales du théorème, l'application moment prend la forme $\mathbf{j}(g, \nu, w) = g \cdot_\sigma (\mu + \nu + \mathbf{J}_{N_1}(w))$. Si le groupe de Lie G est simplement connexe, le cocycle infinitésimal peut être intégré en un cocycle σ et l'application moment $\mathbf{J}$ est bien définie sur un voisinage de Gp invariant sous l'action de G. Plus généralement, l'application moment existe toujours sur un espace de revêtement approprié d'un voisinage d'une orbite de groupe. En utilisant le revêtement universel $\tilde{G}$ du groupe $G = \tilde{G}/H$, l'application moment est bien définie modulo H dans un voisinage $\mathcal{U}$ de Gp.

Comme exemple on considère les équilibres relatifs de systèmes hamiltoniens avec une symétrie $G = \mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ opérant librement. En utilisant les définitions et la construction généralisée aux applications moments avec un cocycle, on voit que pour un cocycle non-trivial, le groupe d'isotropie est conjugué à SO(2). En conséquence, tous les équilibres relatifs sont des ondes rotatives. Si le cocycle s'annule, les équilibres relatifs pour un moment générique sont des ondes planes et seulement les équilibres relatifs avec un moment $\mu = 0$ peuvent roter. Cela s'applique à l'étude de la dynamique des tourbillons dans le plan.

1 Introduction

We consider a finite dimensional connected symplectic manifold $(\mathcal{M}, \omega)$ and a smooth Hamiltonian $H : \mathcal{M} \to \mathbb{R}$ which generates a Hamiltonian vector field X_H on $\mathcal{M}$ via the relationship

$$\omega(p)(X_H(p), v) = \mathrm{D}H(p)v \quad \forall p \in \mathcal{M}\ \forall v \in T_x\mathcal{M}. \tag{1}$$

Suppose that a finite dimensional Lie group G acts properly and symplectically on $\mathcal{M}$ and that the Hamiltonian H is G-invariant. By Noether's theorem, the continuous part of the symmetry group causes the Hamiltonian system to possess conserved quantities, called *momenta* [2, 4]. For any $p \in \mathcal{M}$ there is an open neighbourhood $\mathcal{U}_p$ containing p such that the vector field $p \to \xi_\mathcal{M}(p)$ on $\mathcal{U}_p$ is Hamiltonian for every $\xi \in \mathbf{g} = T_{id}G$ with a Hamiltonian $\mathbf{J}_\xi : \mathcal{U}_p \to \mathbb{R}$. The map $\mathbf{J}_\xi$ can be chosen to be linear in ξ and so defines a *momentum map* $\mathbf{J} : \mathcal{U}_p \to \mathbf{g}^*$ by $\langle \mathbf{J}(p), \xi \rangle = \mathbf{J}_\xi(p)$ where $\langle , \rangle$ denotes the pairing between $\mathbf{g}^*$ and $\mathbf{g}$.
In this note we regard momentum maps which are not infinitesimally equivariant with respect to the coadjoint action and focus on the differences caused by non-trivial cocycles of the momentum map. As in [4] we define the *infinitesimal cocycle* (or 2-cocycle) $\Sigma : \mathbf{g} \times \mathbf{g} \to \mathbb{R}$ by $\Sigma(\xi, \eta) = \mathbf{J}_{[\xi,\eta]}(x) - \{\mathbf{J}_\xi, \mathbf{J}_\eta\}(x)$ for $\xi, \eta \in \mathbf{g}$ and an arbitrary $x \in \mathcal{U}_p$. We can now define a new Σ-dependent action of $\mathbf{g}$ on $\mathbf{g}^*$:

$$\xi_{\mathbf{g}^*,\Sigma}(\mu) = -ad^*_\xi\mu + \Sigma(\xi, \cdot) \quad \text{for} \quad \xi \in \mathbf{g}. \tag{2}$$

By construction, the local momentum map $\mathbf{J}$ on $\mathcal{U}_p$ is infinitesimally equivariant with regard to this action [2, 4].
In this note we generalize the results on the explicit form of Hamilton's equations in local coordinates near a group orbit of Roberts, Wulff and Lamb [7] to systems which have local momentum maps with cocycles. These equations are especially helpful to analyse the stability and bifurcations of relative equilibria, ie group orbits which are invariant under the flow of the Hamiltonian system. Examples where momentum maps with cocycles occur include vortex dynamics and charged particles in magnetic fields [4].

2 The Witt decomposition

In this section we present a Witt decomposition of the tangent space $T_p\mathcal{M}$ for local momentum maps with cocycles (for a similar account see [5]). We denote by $(\,,)$ a G_p-invariant inner product on $T_p\mathcal{M}$ (the orthogonal complement to a subspace W will be denoted by $W^\perp$) and by $W^\omega := \{v \in T_p\mathcal{M} \mid \omega(p)(v,w) = 0 \;\forall w \in W\}$ the orthogonal complement with respect to $\omega(p)$ to a subspace W.

Proposition 1 *Define the following subspaces of* $T_p\mathcal{M}$: $T := T_p(Gp)$, $T_0 := T \cap T^\omega$, $T_1 := T \cap T_0^\perp$, $N_0 := (T + T^\omega)^\perp$ *and* $N_1 := T^\omega \cap T_0^\perp$.

a) We can decompose $T_p\mathcal{M} = T_0 \oplus T_1 \oplus N_0 \oplus N_1$. *Here* $T = T_0 \oplus T_1$ *is the tangent space to the group orbit* Gp *in* p *and* $N := N_0 \oplus N_1$ *is the normal space.*

b) The subspaces $T_0 \oplus N_0$, T_1 *and* N_1 *are symplectic with symplectic forms* $\omega_0 := \omega(p)|_{T_0\oplus N_0}$, $\omega_{T_1} := \omega(p)|_{T_1}$, $\omega_{N_1} := \omega(p)|_{N_1}$ *such that* $\omega(p) = \omega_0 \oplus \omega_{T1} \oplus \omega_{N_1}$.

c) $\omega(p)$ *defines an isomorphism between the subspaces* N_0 *and* T_0^* *which transforms* ω_0 *into the natural symplectic form on* $T_0 \oplus T_0^*$: $\omega_0((u,\nu),(\hat{u},\hat{\nu})) = -\hat{\nu}(u) + \nu(\hat{u})$ $(u, \hat{u} \in T_0,\ \nu, \hat{\nu} \in T_0^*)$.

d) For $\mathbf{J}(p) = \mu$ *denote by* $\mathbf{g}_\mu^\Sigma = \{\xi \in \mathbf{g} \mid \xi_{\mathbf{g}^*,\Sigma}\,\mu = 0\}$ *the isotropy algebra of* μ *with respect to the action (2). Then* $T_0 \simeq \mathbf{g}_\mu^\Sigma/\mathbf{g}_p$. *Moreover* $D\mathbf{J}(p)$ *maps* T_1 *isomorphically onto* $\mathbf{g}/\mathbf{g}_\mu^\Sigma$ *and* ω_{T_1} *onto the symplectic form* ω_μ *on* $\mathbf{g}_{\mathbf{g}^*,\Sigma}\mu \cong \mathbf{g}/\mathbf{g}_\mu^\Sigma$ *which is defined as follows:*

$$\forall\, \xi, \hat{\xi} \in \mathbf{g} \quad \omega_{T_1}(\xi p, \hat{\xi} p) = \mu([\xi,\hat{\xi}]) - \Sigma(\xi,\hat{\xi}) = \omega_\mu(\xi_{\mathbf{g}^*,\Sigma}(\mu), \hat{\xi}_{\mathbf{g}^*,\Sigma}(\mu)).$$

The proof is analogous to the Witt decomposition for Ad^*-equivariant momentum maps used in [7]. Part d) is contained in [2, Cor. IV.5.3] for the case of global momentum maps with cocycles.

3 A local model for $\mathcal{M}$

We begin by constructing a 'local model' for $\mathcal{M}$ following the approach in [7] and then use this to derive the form of Hamilton's equations near a relative equilibrium.
For $\mu \in \mathbf{g}^*$ denote by $\mathbf{n}_\mu^\Sigma$ a G_p-invariant complement to $\mathbf{g}_\mu^\Sigma$ in $\mathbf{g}$, such that $\mathbf{g} \cong \mathbf{g}_\mu^\Sigma \oplus \mathbf{n}_\mu^\Sigma$. Let $\mathrm{ann}(\mathbf{g}_\mu^\Sigma) = \mathrm{ann}_{\mathbf{g}^*}(\mathbf{g}_\mu^\Sigma) = \{\nu \in \mathbf{g}^* \mid \nu(\xi) = 0 \;\forall \xi \in \mathbf{g}_\mu^\Sigma\}$ denote the *annihilator of* $\mathbf{g}_\mu^\Sigma$ *in* $\mathbf{g}^*$ and $\mathrm{ann}(\mathbf{n}_\mu^\Sigma)$ the annihilator of $\mathbf{n}_\mu^\Sigma$. Then we have $\mathbf{g}^* \cong \mathrm{ann}(\mathbf{g}_\mu^\Sigma) \oplus \mathrm{ann}(\mathbf{n}_\mu^\Sigma)$. We will identify $\mathrm{ann}(\mathbf{g}_\mu^\Sigma)$ with $(\mathbf{g}/\mathbf{g}_\mu^\Sigma)^*$ and $(\mathbf{g}_\mu^\Sigma)^*$ with $\mathrm{ann}(\mathbf{n}_\mu^\Sigma)$. Moreover let $\mathbf{m}_\mu^\Sigma$ be a G_p-invariant complement to $\mathbf{g}_p$ in $\mathbf{g}_\mu^\Sigma$ and identify $\mathrm{ann}_{(\mathbf{g}_\mu^\Sigma)^*}(\mathbf{g}_p) \cong (\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^*$, $\mathrm{ann}_{(\mathbf{g}_\mu^\Sigma)^*}(\mathbf{m}_\mu^\Sigma) \cong \mathbf{g}_p^*$.
By the Palais slice theorem [6] a G-invariant neighbourhood $\mathcal{U} \subset \mathcal{M}$ of a group orbit Gp is diffeomorphic to a G-invariant neighbourhood of $p \cong (id, 0)$ in $G \times_{G_p} N$ where we identify $(gg_p^{-1}, g_p w) = (g, w)$ for $g \in G$, $g_p \in G_p$, $w \in N$. Since $N \cong (\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^* \oplus N_1$ for $\mu = \mathbf{J}(p)$, by Proposition 1, the manifold $\tilde{\mathcal{M}}_0 = G \times_{G_p} ((\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^* \oplus N_1) \cong G \times_{G_p} N$ is locally near Gp a model for $\mathcal{M}$. Define a smooth action of $G \times G_p$ on $\tilde{\mathcal{M}} = G \times ((\mathbf{g}_\mu^\Sigma)^* \oplus N_1)$ by

$$(\tilde{g}, g_p) \cdot (g, \zeta, w) = (\tilde{g} g g_p^{-1}, Ad^*_{g_p^{-1}} \zeta, g_p w) \quad \text{for} \quad \tilde{g} \in G, g_p \in G_p, (g,\zeta,w) \in G \times ((\mathbf{g}_\mu^\Sigma)^* \oplus N_1). \tag{3}$$

Define a 2-form $\tilde{\omega}$ on $\tilde{\mathcal{M}}$ by

$$\tilde{\omega}(g,\zeta,w) = \tilde{\omega}_G(g,\zeta) + \tilde{\omega}_\mu(g) + \tilde{\omega}_{N_1}(w) \quad \text{for } (g,\zeta,w) \in \tilde{\mathcal{M}}$$

where

(i) $\tilde{\omega}_G$ is the pullback of the natural symplectic form ω_G on $T^*G \cong G \times \mathbf{g}^*$:
$\omega_G(g,\zeta)((g\xi,\alpha),(g\hat{\xi},\hat{\alpha})) = \hat{\alpha}(\xi) - \alpha(\hat{\xi}) + \zeta([\xi,\hat{\xi}]) \ \ \forall g \in G \ \ \forall \zeta,\alpha,\hat{\alpha} \in \mathbf{g}^* \ \ \forall \xi,\hat{\xi} \in \mathbf{g}$
by the map $(g,\zeta,w) \mapsto (g, i_\mu \zeta)$ where the imbedding $i_\mu : (\mathbf{g}_\mu^\Sigma)^* \to \mathbf{g}^*$ is induced by the complement $\mathbf{n}_\mu^\Sigma$ to $\mathbf{g}_\mu^\Sigma$ in $\mathbf{g}$.

(ii) $\tilde{\omega}_\mu$ is the pullback of the 2-form $\Omega_\mu(g)(g\xi, g\hat{\xi}) = \mu([\xi,\hat{\xi}]) - \Sigma(\xi,\hat{\xi}) \ \forall \xi,\hat{\xi} \in \mathbf{g}$, by the map $(g,\zeta,w) \mapsto g$.

(iii) $\tilde{\omega}_{N_1}$ is the pullback of the symplectic form ω_{N_1} on N_1 by $(g,\zeta,w) \mapsto w$.

Then the 2-form $\tilde{\omega}$ is a $G \times G_p$-invariant symplectic form on a G-invariant neighbourhood of $G \times \{(0,0)\}$ in $\tilde{\mathcal{M}}$. Let $\mathbf{J}_{N_1} : N_1 \to \mathbf{g}_p^*$ be the momentum map for the symplectic action of G_p on N_1 (often called vibrational angular momentum [7]). Then, as in [7], the map $\psi : \tilde{\mathcal{M}} \to \mathbf{g}_p^*$, $\psi(g,\zeta,w) = \mathbf{J}_{N_1}(w) - P_{\mathbf{g}_p^*}\zeta$, where $P_{\mathbf{g}_p^*}$ is the projection onto $\mathrm{ann}_{(\mathbf{g}_\mu^\Sigma)^*}(\mathbf{m}_\mu^\Sigma)$ with kernel $\mathrm{ann}(\mathbf{g}_p)$, is a momentum map for the free symplectic G_p-action (3) on $\tilde{\mathcal{M}}$, and $\psi^{-1}(0)/G_p = \tilde{\mathcal{M}}_0$. By Proposition 1 the restriction of the form $\tilde{\omega}$ on $T_{(id,0,0)}\tilde{\mathcal{M}}_0$ is isomorphic to $\omega(p)$. The relative Darboux theorem (see [7] for details) therefore implies the following theorem which in the case of global momentum maps is contained in [1, 3] and which has been proved independently in [5]:

Theorem 1 *There exists a G-equivariant symplectomorphism Φ from a neighbourhood $\tilde{\mathcal{U}}_0$ of $G \times_{G_p} \{0\}$ in $G \times_{G_p} N$ onto a neighbourhood $\mathcal{U}$ of Gp in $\mathcal{M}$.*

4 Hamiltonian equations near relative equilibria

The pullback of the G-invariant Hamiltonian function H on $\mathcal{M}$ under the symplectomorphism $\Phi : \tilde{\mathcal{U}}_0 \to \mathcal{U}$ is a function $h := H \circ \Phi$ on $\mathcal{U} \subset G \times N$ which is independent of $g \in G$ and depends only on the variables in the normal space $(\nu, w) \in (\mathbf{g}_\mu^\Sigma/\mathbf{g}_p)^* \oplus N_1 \cong N$. Extend h to a function $\tilde{h} : (\mathbf{g}_\mu^\Sigma)^* \oplus N_1 \to \mathbb{R}$ by defining $\tilde{h}(\zeta,w) = h(P_{(\mathbf{g}_\mu/\mathbf{g}_p)^*}\zeta, w)$ where $P^*_{(\mathbf{g}_\mu/\mathbf{g}_p)} = id - P_{\mathbf{g}_p^*}$. By (1) the system of Hamiltonian equations on $\tilde{\mathcal{U}} \subset \tilde{\mathcal{M}} = G \times ((\mathbf{g}_\mu^\Sigma)^* \oplus N_1)$ is determined by the equation

$$\tilde{\omega}(g,\zeta,w)((\dot{g},\dot{\zeta},\dot{w}),(\hat{g},\hat{\zeta},\hat{w})) = D_\zeta \tilde{h}(\zeta,w)\hat{\zeta} + D_w \tilde{h}(\zeta,w)\hat{w} \tag{4}$$

where $g \in G$, $\dot{g},\hat{g} \in g\mathbf{g} \cong T_gG$, $\zeta,\dot{\zeta},\hat{\zeta} \in (\mathbf{g}_\mu^\Sigma)^*$, $w,\dot{w},\hat{w} \in N_1$, and

$$\begin{aligned}\tilde{\omega}(g,\zeta,w)((\dot{g},\dot{\zeta},\dot{w}),(\hat{g},\hat{\zeta},\hat{w})) \;=\; & i_\mu\hat{\zeta}(g^{-1}\dot{g}) - i_\mu\dot{\zeta}(g^{-1}\hat{g}) + (i_\mu\zeta + \mu)([g^{-1}\dot{g}, g^{-1}\hat{g}]) \\ & -\Sigma(g^{-1}\dot{g}, g^{-1}\hat{g}) + \omega_{N_1}(\dot{w},\hat{w}).\end{aligned} \tag{5}$$

Denoting by $P_{\mathbf{g}_\mu^\Sigma}$ the projection onto $\mathbf{g}_\mu^\Sigma$ with kernel $\mathbf{n}_\mu^\Sigma$ Equations (4), (5) become:

$$P_{\mathbf{g}_\mu^\sigma}(g^{-1}\dot{g}) = D_\zeta h(\zeta,w) \tag{6}$$

$$i_\mu\dot{\zeta} = ad^*_{g^{-1}\dot{g}}(i_\mu\zeta + \mu) - \Sigma(g^{-1}\dot{g},\cdot) \tag{7}$$

$$J_{N_1}^T\dot{w} = D_w h(\zeta,w) \ \Leftrightarrow\ \dot{w} = J_{N_1} D_w h(\zeta,w) \tag{8}$$

Denoting $g^{-1}\dot{g} = \xi + \eta \in \mathbf{g}_\mu^\Sigma \oplus \mathbf{n}_\mu^\Sigma \cong \mathbf{g}$ we can write equation (7) as follows:

$$\dot{\zeta} = P_{\mathrm{ann}(\mathbf{n}_\mu^\Sigma)}(ad^*_{\xi+\eta}(i_\mu\zeta + \mu) - \Sigma(\xi+\eta,\cdot)) \tag{9}$$

$$0 = P_{\mathrm{ann}(\mathbf{g}_\mu^\Sigma)}(ad^*_{\xi+\eta}(i_\mu\zeta + \mu) - \Sigma(\xi+\eta,\cdot)) \tag{10}$$

The next proposition ensures that equation (10) has a solution for given $\xi \in \mathbf{g}_\mu^\Sigma$.

Proposition 2 *For $\xi \in \mathbf{g}_\mu^\Sigma$ and $i_\mu \zeta$ sufficiently close to 0 in ann$(\mathbf{n}_\mu^\Sigma)$ equation (10) has a unique solution $\eta = \eta_\mu(\xi, \zeta) \in \mathbf{n}_\mu^\Sigma$ which is linear in ξ.*

This proposition is proved by the implicit function theorem, as in [7]. If $\mathbf{n}_\mu^\Sigma$ is a $\mathbf{g}_\mu^\Sigma$- invariant complement of $\mathbf{g}_\mu^\Sigma$ in $\mathbf{g}$ (with respect to the usual adjoint action) then $\eta_\mu \equiv 0$. Momenta μ which allow such complements are called *split*.
Now we can define for ζ sufficiently close to 0 in $(\mathbf{g}_\mu^\Sigma)^*$ the linear map $j_\mu(\zeta) : \mathbf{g}_\mu^\Sigma \to \mathbf{g}$ by $j_\mu(\zeta)\xi = \xi + \eta_\mu(\xi, \zeta)$. Using Proposition 2 and (6) – (10) we obtain our main theorem:

Theorem 2 *In the coordinates (g, ν, w) from Theorem 1 and with $\zeta = \nu + \mathbf{J}_{N_1}(w)$ the Hamiltonian equations in a G-invariant neighbourhood of Gp are:*

$$\begin{aligned} \dot{g} &= g j_\mu(\zeta) D_\zeta \tilde{h}(\zeta, w) &(11)\\ i_\mu \dot{\zeta} &= -(j_\mu(\zeta) D_\zeta \tilde{h}(\zeta, w))_{\mathbf{g}^*, \Sigma}(\mu + i_\mu \zeta) &(12)\\ \dot{w} &= J_{N_1} D_w \tilde{h}(\zeta, w) &(13) \end{aligned}$$

If $\eta_\mu \equiv 0$, for example if μ is split, then equation (12) becomes $\dot{\zeta} = ad^*_{D_\zeta \tilde{h}(\zeta, w)} \zeta$.

The equation for ν can be obtained by projecting (12) onto $(\mathbf{g}_\mu^\Sigma / \mathbf{g}_p)^* = \mathrm{ann}_{(\mathbf{g}_\mu^\Sigma)^*}(\mathbf{g}_p)$ as in [7].

5 The momentum map near group orbits

If the momentum map $\mathbf{J}$ exists on the G-invariant neighbourhood $\mathcal{U}$ of Gp then it is equivariant for the action $g \cdot_\sigma \mu := Ad^*_{g^{-1}} \mu + \sigma(g)$ on $\mathbf{g}^*$. Here the *1-cocycle* $\sigma : G \to \mathbf{g}^*$ is defined by $\sigma(g) = \mathbf{J}(gp) - Ad^*_{g^{-1}} \mathbf{J}(p)$ for $g \in G$ and an arbitrary $p \in \mathcal{U}$, and $\Sigma = D\sigma_\eta(id)\xi$ (see [2, 4] for details). Under this assumption it is straightforward to check, cf. [1, 3, 7], that the momentum map $\mathbf{j}(g, \nu, w)$ in the coordinates (g, ν, w) takes the form

$$\mathbf{j}(g, \nu, w) = g \cdot_\sigma (\mu + \nu + \mathbf{J}_{N_1}(w)). \tag{14}$$

If G is simply connected then this assumption is satisfied. To see this note that, by Theorem 1, $\mathcal{U}$ is symplectomorphic to a G-invariant neighbourhood of $G \times \{0, 0\}$ in $\tilde{\mathcal{M}}_0 = G \times_{G_p} ((\mathbf{g}_\mu^\Sigma / \mathbf{g}_p)^* \oplus N_1)$. Since the symplectic manifold $\tilde{\mathcal{M}} = G \times ((\mathbf{g}_\mu^\Sigma)^* \oplus N_1)$ is simply connected, $\tilde{\mathbf{j}}(g, \zeta, w) = g \cdot_\sigma (\mu + \zeta)$ is a global momentum map for the G-action on $\tilde{\mathcal{M}}$ [4]. It therefore descends to the momentum map (14) on $\tilde{\mathcal{M}}_0$ and $\mathcal{U}$.
More generally, momentum maps always exist on appropriate covering spaces of neighbourhoods of group orbits: Let $\tilde{G}$ be the universal covering of G such that $\tilde{G}/H = G$ where the fundamental group H of $\tilde{G}$ is a discrete subgroup of the center of $\tilde{G}$. Then, as we have just seen, the manifold $\tilde{G} \times_{G_p} N$ has a global momentum map $\mathbf{j}(g, \nu, w)$, defined as in (14). On $\mathcal{U} \cong \tilde{G}/H \times_{G_p} N$ this momentum map is well-defined modulo H.

6 Applications

6.1 Central extensions and bifurcations of relative equilibria

Suppose $\mathbf{J} : \mathcal{U}_p \to \mathbf{g}^*$ is a local momentum map with infinitesimal cocycle Σ. The *central extension of* $\mathbf{g}$ is the Lie algebra $\mathbf{g}' = \mathbf{g} \oplus \mathbb{R}$ with commutator bracket defined by $[(\xi, a), (\eta, b)]_{\mathbf{g}'} = ([\xi, \eta]_{\mathbf{g}}, \Sigma(\xi, \eta))$
where $\xi, \eta \in \mathbf{g}$, $a, b \in \mathbb{R}$ and $[\,,]_{\mathbf{g}}$ denotes the commutator bracket on $\mathbf{g}$. Writing $\widetilde{ad}_{(\xi,a)}(\eta, b) = [(\xi, a), (\eta, b)]_{\mathbf{g}'}$ the coadjoint action of $\mathbf{g}'$ on $\mathbf{g}'^* = \mathbf{g}^*$ becomes $\widetilde{ad}^*_{(\xi,a)}(\mu, \alpha) = (ad^*_\xi \mu + \alpha \Sigma(\xi, \cdot), 0)$. For $\mu' := (\mu, -1) \in (\mathbf{g}')^*$ we get $\mathbf{g}'_{(\mu,-1)} = \{(\xi, a) \in \mathbf{g}' | \widetilde{ad}^*_{(\xi,a)}(\mu, -1) = 0\} = \mathbf{g}_\mu^\Sigma \oplus \mathbb{R}$. If we define the action of $\mathbf{g}'$ on $\mathcal{M}$ by $(\xi, a)_{\mathcal{M}}(x) = \xi_{\mathcal{M}}(x)$ then the induced local momentum map $\mathbf{J}' : \mathcal{U}_p \to \mathbf{g}'^*$ given by

$$\langle \mathbf{J}'(z), (\xi, a) \rangle_{\mathbf{g}'^*, \mathbf{g}'} = \langle (\mathbf{J}(z), -1), (\xi, a) \rangle_{\mathbf{g}'^*, \mathbf{g}'} = \langle \mathbf{J}(z), \xi \rangle_{\mathbf{g}^*, \mathbf{g}} - a$$

is infinitesimally equivariant with respect to the coadjoint action of $\mathbf{g}'$ on $\mathbf{g}'^*$, see [4].
Adding $\dot{\alpha} = 0$ to (12) the corresponding differential equations for $\zeta' = (\zeta, \alpha) \in (\mathbf{g}')^*$ contain no cocycle and have the same form as the corresponding equations in [7]. We can therefore apply results derived for coadjoint symmetry group actions on momentum space to study bifurcations and stability of relative equilibria. For example, we see that if μ is minimal, i.e., $\mathbf{g}_\mu^\Sigma$ has minimal dimension for $\mu \in \mathbf{g}^*$, then also $\mathbf{g}'_\mu$ has minimal dimension and therefore $\mathbf{g}'_\mu$ is abelian and $\dot{\zeta} = 0$. This yields persistence and stability results for relative equilibria with minimal momenta, cf. [2, 7]. Moreover we see that minimality is a generic condition in $\mathbf{g}^*$ and that for abelian groups G every $\mu \in \mathbf{g}^*$ is minimal, as for coadjoint actions [4].

6.2 Example: Hamiltonian systems with Euclidean symmetry

As an example we consider relative equilibria of Hamiltonian systems with symmetry $G = \mathrm{SE}(2)$ acting freely. Recall that $\mathrm{SE}(2) = \mathrm{SO}(2) \ltimes \mathbb{R}^2$ is the special Euclidean group of the plane. Cocycles are given by scalar multiples of $\sigma(\phi, s) = (-\frac{1}{2}|s|^2, s_2, -s_1)$ [4]. One computes that for non-vanishing cocycle the isotropy G_μ^σ for the action $g \cdot_\sigma \mu := Ad^*_{g^{-1}}\mu + \sigma(g)$ is conjugate to $\mathrm{SO}(2)$ for all $\mu \in \mathbf{g}^*$. As a consequence every μ is minimal and $\dot{\nu} \equiv 0$. Since the drift velocity ξ_p of a relative equilibrium p with momentum $\mu = \mathbf{J}(p)$ satisfies $\xi_p \in \mathbf{g}_\mu^\Sigma$ by Theorem 2, and since $\mathbf{g}_\mu^\Sigma \equiv \mathrm{so}(2)$, we see that relative equilibria are rotating waves whenever the cocycle σ does not vanish. For a vanishing cocycle relative equilibria with generic momentum $\mu \in \mathbf{g}^*$ are travelling waves, i.e. satisfy $\xi_p \in \mathbb{R}$, and only relative equilibria with momentum $\mu = 0$ can rotate [7]. This has implications for the dynamics of vortices on the plane. Here the cocycle vanishes if the total vorticity is zero. If the vorticities do not sum up to zero relative equilibria of point vortices have to be rotating waves.

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