STABILITY OF HAMILTONIAN RELATIVE EQUILIBRIA BY ENERGY METHODS

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For a compact group G a relative equilibrium of a G-invariant Hamiltonian is orbitally Liapounov stable (G-stable) if the Hessian of an augmented Hamiltonian at the corresponding critical point is definite when restricted to the symplectic normal space. This is no longer true in general for proper actions of noncompact groups, essentially because the orbit space of the coadjoint action of the group need not be Hausdorff. This paper gives a summary of our results on the stability of Hamiltonian relative equilibria. It presents a more general stability criterion that applies to relative equilibria with 'tame' angular velocities, and also gives a result that sharpens G-stability to A-stability, where A is a subset of G that depends only on the momentum of the relative equilibrium.

1 Introduction

Consider a Hamiltonian system

$$\dot{w} = J\mathrm{D}_w H(w)$$

on $M = \mathbb{R}^{2n}$ where $J = \begin{pmatrix} 0 & \mathrm{id} \\ -\mathrm{id} & 0 \end{pmatrix}$ and $H : M \to \mathbb{R}$ is a smooth Hamiltonian. Let 0 be an equilibrium of this system, ie $J\mathrm{D}_w H(0) = 0$. It is well-known and easily checked that the linearization $L = J\mathrm{D}^2_w H(0)$ of the Hamiltonian system at 0 is infinitesimally symplectic, ie satisfies $JL = -L^T J$. So if λ is an eigenvalue of L then so is $-\lambda$. As a consequence, a necessary condition for the equilibrium to be stable is that $\Re\,\mathrm{spec}(L) = 0$. This means that nonlinear stability of an equilibrium of a Hamiltonian system can not be inferred from linear information, in contrast to the situation for general non-Hamiltonian systems. A simple sufficient condition for stability uses the conservation of energy: $H(w(t)) = H(w(0))$ for all solutions $w(t)$ of the Hamiltonian system.

If $D^2H(0)$ is definite at 0 then 0 is a minimum or maximum of H and therefore Liapounov stable.

The aim of this paper, which summarises a longer paper[13], is to generalize this approach to relative equilibria of Hamiltonian systems with symmetry.

2 Hamiltonian systems with symmetry

Let M be a finite dimensional symplectic manifold, G a Lie group which acts properly and (for simplicity only) *freely* on M, and ω a G-invariant symplectic form on M. Let $H : M \to \mathbb{R}$ be a G-invariant Hamiltonian, and

$$\dot{x} = f_H(x) \tag{1}$$

the Hamiltonian system defined by $\omega_x(f_H(x), v) = DH(x)v$ for $v \in T_xM$. Let $\mathbf{g}$ denote the Lie algebra of G and $\mathbf{g}^*$ its dual. By Noether's Theorem for each continuous symmetry $\xi \in \mathbf{g}$ locally there is a conserved quantity $\mathbf{J}(\xi)$ of (1) which is linear in ξ, so that $\mathbf{J}$ maps into $\mathbf{g}^*$ (see eg Marsden and Ratiu[7]). We will assume that $\mathbf{J}$ exists globally on M and is equivariant with respect to the usual coadjoint action of G on $\mathbf{g}^*$, given by $g\mu := (\mathrm{Ad}_g^*)^{-1}\mu$ where $\mathrm{Ad}_g : \mathbf{g} \to \mathbf{g}$ defined by $\mathrm{Ad}_g\eta = g\eta g^{-1}$ is the adjoint action. We also define the adjoint action of $\mathbf{g}$ on itself by $\mathrm{ad}_\xi\eta = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\exp(t\xi)}\eta|_{t=0} = [\xi, \eta]$ and the dual coadjoint action of $\mathbf{g}$ on $\mathbf{g}^*$ by $\xi\mu = -\mathrm{ad}_\xi^*\mu$. A momentum value μ is defined to be *regular* if the coadjoint orbit of every nearby point has the same dimension as $G\mu$. The set of regular momenta is open and dense in $\mathbf{g}^*$ and if μ is regular then the coadjoint isotropy algebra $\mathbf{g}_\mu = \{\xi \in \mathbf{g} : \mathrm{ad}_\xi^*\mu = 0\}$ is Abelian.

3 Relative equilibria and skew product equations

A group orbit Gp is called a *relative equilibrium* if it is an equilibrium in the space of group orbits, ie if the solution of (1) with initial value p satisfies $x(t;p) \in Gp$ for all $t \in \mathbb{R}$. It is easily checked that Gp is a relative equilibrium if and only if there exists $\xi_p \in \mathbf{g}$, called the *drift velocity* of the relative equilibrium, such that $\xi_p p := \frac{\mathrm{d}}{\mathrm{d}t}\exp(t\xi_p)p|_{t=0} = f_H(p)$. Let U be a G-invariant neighbourhood of Gp in M and let $N \subset T_pM$ be a normal space to the group orbit $\mathbf{g}p$ at p, so that $\mathbf{g}p \oplus N = T_pM$. Using Palais slice coordinates[11] we can write every $x \in U$ as $x \simeq (g, v)$ where $g \in G$ and $v \in N$, see Figure 1. In these coordinates (1) takes the form

$$\dot{g} = gf_G(v), \quad \dot{v} = f_N(v). \tag{2}$$

This was proved for general systems by Krupa[3] and Fiedler et al[2]. The first equation describes the motion along group orbits, and the second equation the shape dynamics.

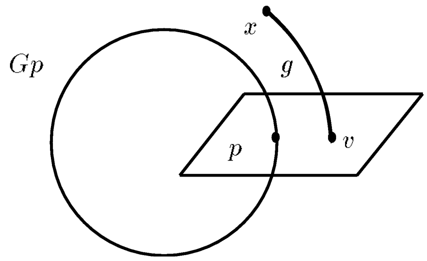

Figure 1. Palais coordinates near Gp

4 Hamiltonian relative equilibria

Examples of Hamiltonian relative equilibria include rotating molecules[9] and rotating and translating rigid bodies in fluids[4]. In the first case the effective symmetry group is compact since without loss of generality the centre of mass can be assumed to be fixed, but in the second case the symmetry group is the noncompact special Euclidean group $\mathrm{SE}(3) = \mathrm{SO}(3) \triangleright \mathbb{R}^3$ of rotations and translations of three space.

Stability results for relative equilibria with regular momenta have been obtained by Arnold[1] and Libermann and Marle[6], and for relative equilibria with arbitrary momenta and compact symmetry groups G (or 'almost compact' symmetry groups, see below) by Montaldi[8], Lerman and Singer[5] and Ortega and Ratiu[10]. These results do not apply to rigid bodies in fluids with non-regular momenta. To cover these Leonard and Marsden[4] presented some stability results for noncompact groups which are semidirect products of vector spaces and compact groups. We have improved these results and developed a stability theory via energy methods for general proper noncompact group actions. The key ideas are summarized below. Our main tool in this paper are the bundle equations (3) for Hamiltonian systems that we derived with Lamb[14]. We recall these in the next section.

5 Hamiltonian systems near group orbits

Let $\mu = \mathbf{J}(p)$ be the momentum of $p \in M$, let G_μ be its isotropy with respect to the coadjoint group action and $\mathbf{g}_\mu$ be its Lie algebra. In this paper we assume that μ is *split*, ie there is a G_μ^0-invariant complement to $\mathbf{g}_\mu$ in $\mathbf{g}$, where G_μ^0 is the identity component of G_μ. The slice N to Gp at p from Section 3 has the Witt decomposition $N = N_0 \oplus N_1$. The space N_0 is isomorphic to $\mathbf{g}_\mu^*$ and can be identified with an affine subspace of $\mathbf{g}^*$ through μ which is transverse to $G\mu$ at μ. The space N_1 is a slice to $G_\mu p$ in the momentum level set $\mathbf{J}^{-1}(\mu)$. The symplectic form on T_pM restricts to a symplectic form on N_1, which we therefore call the *symplectic normal space* at p. See eg Roberts et al[14] for further details. Let J_{N_1} denote the skew-symmetric matrix that defines the symplectic form on N_1. We have the following theorem:

Theorem 1 [14] *For Hamiltonian systems and $\mu = \mathbf{J}(p)$ split, the bundle equations (2) near Gp take the form:*

$$\dot{g} = g\mathrm{D}_\nu h(\nu, w), \quad \dot{\nu} = \mathrm{ad}^*_{\mathrm{D}_\nu h(\nu,w)}\nu, \quad \dot{w} = J_{N_1}\mathrm{D}_w h(\nu, w) \tag{3}$$

where $h(\nu, w)$ is the function on the slice $N_0 \oplus N_1$ obtained by restricting the Hamiltonian H. If μ is regular then $\mathbf{g}_\mu$ is Abelian and so $\dot{\nu} \equiv 0$.

The first equation describes the motion of the body frame, the second equation the motion of the momenta in body coordinates and the last equation the shape dynamics. We refer to the $(\dot{\nu}, \dot{w})$ equations as the *slice equations.*

6 Hamiltonian relative equilibria and slice equations

An orbit Gp is a relative equilibrium if and only if p is a critical point of the Hamiltonian in the comoving frame $H_{\xi_p}(x) = H(x) - \mathbf{J}_{\xi_p}(x)$ and if and only if 0 is an equilibrium of the slice equations on $N = N_0 \oplus N_1$. For any subset $A \subset G$ we say that a relative equilibrium Gp is A-*stable* if for all x_0 close to p the trajectory $x(t; x_0)$ is close to Ap for all $t \in \mathbb{R}$. A relative equilibrium Gp is G-stable if and only if 0 is a stable equilibrium of the slice equations.

We see from Theorem 1 that for μ regular $\nu \in N_0$ can be treated as a parameter and so 0 is a stable equilibrium of the slice equations if it is a stable equilibrium of the ν-parametrised Hamiltonian system $\dot{w} = J_{N_1}\mathrm{D}_w h(\nu, w)$ on N_1. This is guaranteed if $\mathrm{D}^2_w h(0)$ is definite, as we saw in Section 1. In this way we recover the stability results of Arnold[1] and Libermann and Marle[6].

For general μ, if all orbits of the $\dot{\nu}$-equation lie in spheres around 0 in N_0 then the dynamics on N_0 is always stable and energy methods only have to be used on N_1 to establish the stability of 0 for the slice equations. This assumption is satisfied if there is a G_μ-invariant inner product on $\mathbf{g}_\mu^*$. In this situation definiteness of $\mathrm{D}^2_w h(0)$ again implies stability. Since $\mathrm{D}^2_w h(0) = \mathrm{D}^2 H_{\xi_p}(p)|_{N_1}$ [13] we recover the energy-momentum method of Ortega and Ratiu[10] and Lerman and Singer[5].

7 G-stability of Hamiltonian relative equilibria

Now we will show how to obtain G-stability for Hamiltonian relative equilibria in the case of a general noncompact symmetry group G. From the equation $\dot{\nu} = \mathrm{ad}^*_{\mathrm{D}_\nu h}\nu$ of (3) we see that $\nu(t) \in G^0_\mu \nu_0$, which just rephrases the statement that the coordinates ν are the momenta in the body frame moving with velocity $\mathrm{D}_\nu h$.

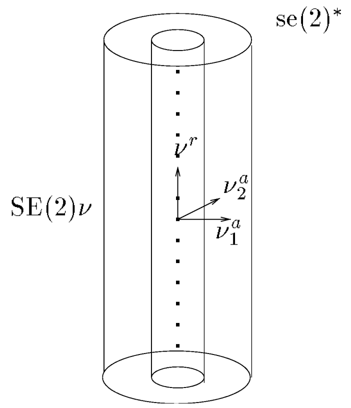

Figure 2. Coadjoint orbits for $G = \mathrm{SE}(2)$

As a motivating example consider the special Euclidean group $G = \mathrm{SE}(2) = \mathrm{SO}(2) \triangleright \mathbb{R}^2$ of the plane and $\mu = 0$. Let $\nu = (\nu^r, \nu^a)$ with $\nu^r \in \mathrm{so}(3)^*$ the angular momentum and $\nu^a = (\nu^a_1, \nu^a_2) \in (\mathbb{R}^2)^*$ the linear momentum. Then the coadjoint group orbits $G\nu$ are either cylinders with axes along the ν^r-axis, or points on the ν^r-axis, see Figure 2. We conclude that the linear momentum $\nu^a(t)$ is always bounded.

We have $\nu^a_i(t) = \nu(t)(\xi^a_i)$, $i = 1, 2$, where ξ^a_i is the i-th component of the translational part ξ^a of the velocity $\xi = (\xi^r, \xi^a) \in \mathrm{so}(2) \oplus \mathbb{R}^2$. We call the infinitesimal translations $\xi^a \in \mathbf{t}_\mu := \mathbb{R}^2$ *tame* because $(G_\mu \nu)(\xi^a)$ is bounded for any $\nu \in \mathbf{g}^*_\mu$, and so $\nu(t)(\xi^a)$ is bounded for any solution of the equation $\dot{\nu} = \mathrm{ad}^*_{\mathrm{D}_\nu h}\nu$. More generally, for any Lie group G and split momentum μ we call $\xi \in \mathbf{g}_\mu$ *tame* if $G^0_\mu \xi$ is bounded. We denote the vector space of all tame velocities for μ by $\mathbf{t}_\mu$.

In the SE(2)-example we see that $\nu^r(t) \in \mathbf{w}^*_\mu := \mathrm{ann}_{\mathbf{g}^*_\mu}(\mathbf{t}_\mu)$ can grow unboundedly along the ν^r-axis of the cylinder. Here $\mathrm{ann}_{\mathbf{g}^*_\mu}(\mathbf{t}_\mu)$ denotes the annihilator of $\mathbf{t}_\mu$ in $\mathbf{g}^*_\mu$. The vector space $\mathbf{w}^*_\mu$ defined in this way for general groups G is called the space of *wild momenta*. We have the following theorem:

Theorem 2 [13] *The relative equilibrium Gp is G-stable if a) its drift velocity ξ_p is tame and b) $\mathrm{D}^2 H_{\xi_p}(p)|_{\mathbf{w}_\mu^* \oplus N_1}$ is definite.*

The tameness of ξ_p is necessary to get a critical point of the Hamiltonian h on $\mathbf{w}_\mu^* \oplus N_1$. If this is satisfied then definiteness of $\mathrm{D}^2 h|_{\mathbf{w}_\mu^* \oplus N_1}$ yields stability of the equilibrium 0 of the slice equations on $N_0 \oplus N_1$ and the equality $\mathrm{D}^2 h(0) = \mathrm{D}^2 H_{\xi_p}(p)|_N$ (see Patrick et al[13]) gives the result. The theorem generalizes the energy-momentum method of Lerman and Singer[5] and Ortega and Ratiu[10] to general noncompact groups. We show in Patrick et al[13] that in fact tameness of ξ_p is a *necessary* condition for any energy-momentum or energy-Casimir method to apply.

Notice that in the SE(2) example instability on $N_0 \cong \mathbf{g}_\mu^*$ is only possible along the ν^r-axis. This is the set of momenta ν which cannot be separated from 0 by open neighbourhoods in the space of group orbits $\mathbf{g}^*/G$. In other words, instability is possible exactly where $\mathbf{g}^*/G$ is not Hausdorff. This is no coincidence. Indeed, the stability theory described in Patrick et al[13] uses this as its starting point and the generalised energy-momentum criteria are deduced from topological stability results on non-Hausdorff topological spaces.

8 A-stability of Hamiltonian relative equlibria

In this section we show that a G-stable relative equilibrium of (1) is actually A-stable for A a certain subset of G that depends only on μ. Again for simplicity we assume that $\mu = \mathbf{J}(p)$ is split. Then by (3) the drift of nearby solutions is governed by the equation $\dot{g} = g\mathrm{D}_\nu h(\nu, w)$, and since $\mathrm{D}_\nu h(\nu, w) \in \mathbf{g}_\mu$ we have $g(t) \in g(0)G_\mu$. So the drift evolution $g(0)^{-1}g(t)$ is in the momentum isotropy subgroup G_μ^0. Since any element x close to p has the bundle coordinates $x \simeq (g, \nu, w)$ with g near id, and ν, w small we get the following theorem:

Theorem 3 [13] *Let Gp be a G-stable relative equilibrium with split momentum $\mu = \mathbf{J}(p)$. Then Gp is A-stable where $A = \bigcup_{g\in W} gG_\mu^0 g^{-1}$ for any neighbourhood W of $\mathrm{id} \in G$.*

This result can be improved further by decomposing $G_\mu^0 = L_\mu K_\mu$ where K_μ is a subgroup for which there exists a K_μ-invariant inner product on $\mathbf{g}^*$. This can always be done using the Levi decomposition, for which K_μ is a maximal compact subgroup of G_μ^0, but some case K_μ can be chosen to be larger than this. If G_μ^0 is decomposed in this way then the subset A defined in the theorem can be replaced by $A = L_\mu^W K_\mu$ where $L_\mu^W = \bigcup_{g\in W} gL_\mu g^{-1}$ for any neighbourhood W of $\mathrm{id} \in G$. In particular, if there exists a G_μ^0-invariant inner product on $\mathbf{g}^*$ then a G-stable relative equilibrium with momentum μ is

G^0_μ-stable, generalising results of Patrick[12], Lerman and Singer[5] and Ortega and Ratiu[10].

As an example consider a rigid body in a fluid with coincident centres of mass and buoyancy[4]. Then the symmetry group G is the special Euclidean group $G = \mathrm{SE}(3) = \mathrm{SO}(3) \triangleright \mathbb{R}^3$ of three space. If μ is regular then $G^0_\mu = \mathrm{SO}(2) \times \mathbb{R}$. A relative equilibrium Gp with this momentum $\mu = \mathbf{J}(p)$ typically translates along and rotates around the SO(2)-axis, which we will assume is the x_1-axis. Denote rotations about the x_1-axis by $\mathrm{SO}(2)_1$ and translations along the x_1-axis by $\mathbb{R}_1$. Then $G^0_\mu = \mathrm{SO}(2)_1 \times \mathbb{R}_1 = \mathbb{R}_1\mathrm{SO}(2)_1$ and we can take $K_\mu = \mathrm{SO}(2)_1$ and $L_\mu = \mathbb{R}_1$. The relative equilibrium is G-stable if $\mathrm{D}^2 H_{\xi_p}(p)|_{N_1}$ is definite. If it is G-stable then by Theorem 3 and the following paragraph it is also A-stable with $A = \mathrm{SO}(2)_1 \times A^a$ where $A^a = \bigcup_{R \in W} R\mathbb{R}_1$ for any neighbourhood W of id $\in \mathrm{SO}(3)$. The set A^a is a cone around the a_1-axis in the group of translations $\mathbb{R}^3$, see Figure 3. The cone can be made arbitrarily 'narrow' by choosing W sufficiently small, and trajectories will remain near the resulting set Ap by choosing the initial conditions sufficiently close to p. Leonard and Marsden[4] observed in numerical simulations that nearby solutions indeed drift within such a cone.

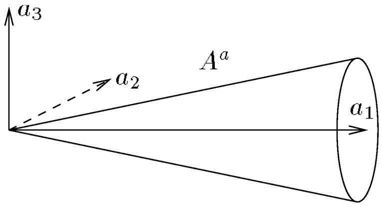

Figure 3. The cone A^a

9 Conclusion

In this paper we have given a summary of our results on the stability of Hamiltonian relative equilibria. In contrast to our forthcoming article[13] we have restricted attention to the simplest possible cases and results, hopefully getting across the main ideas at the expense of generality and optimality. In subsequent publications we will present the general theory and apply them to a variety of examples.

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References

1. V. I. Arnold. *Mathematical Methods of Classical Mechanics.* Springer-Verlag, 1978.
2. B. Fiedler, B. Sandstede, A. Scheel, and C. Wulff. Bifurcation from relative equilibria to noncompact group actions: Skew products, meanders, and drifts. *Doc. Math. J. DMV* **1** : 479–505, 1996.
3. M. Krupa. Bifurcations of relative equilibria. *SIAM J. Math. Anal.* **21** : 1453–1486, 1990.
4. N.E. Leonard and J.E. Marsden. Stability and drift of underwater vehicle dynamics: mechanical systems with rigid motion symmetry. *Physica D* **105** : 130–162, 1997.
5. E. Lerman and S.F. Singer. Stability and persistence of relative equilibria at singular values of the moment map. *Nonlinearity* **11** : 1637–1649, 1998.
6. P. Libermann and C.-M. Marle. *Symplectic Geometry and Analytical Mechanics.* Reidel, Dordrecht, Holland, 1987.
7. J.E. Marsden and T.S. Ratiu. *Introduction to Mechanics and Symmetry.* Springer-Verlag, New York, Berlin, Heidelberg, 1994.
8. J. Montaldi. Persistence and stability of relative equilibria. *Nonlinearity* **10** : 449–466, 1997.
9. J. Montaldi and R.M. Roberts. Relative equilibria of molecules. *J. Nonlin. Sci.* **9** : 53–88, 1999.
10. J.-P. Ortega and T.S. Ratiu. Stability of Hamiltonian relative equilibria. *Nonlinearity* **12** : 693–720, 1999.
11. R.S. Palais. On the existence of slices for actions of noncompact Lie groups. *Ann. of Math.* **73** : 295–323, 1961.
12. G. Patrick. Relative equilibria in Hamiltonian systems: the dynamic interpretation of nonlinear stability on a reduced phase space. *J. Geom. Phys.* **9** : 111–119, 1992.
13. G. Patrick, R.M. Roberts and C. Wulff. Stability of Poisson equilibria and Hamiltonian relative equilibria by energy methods. *In preparation*, 2001.
14. R.M. Roberts, C. Wulff and J.S.W. Lamb. Hamiltonian systems near relative equilibria. *J. Differential Equations*, to appear.