Transitions from relative equilibria to relative periodic orbits

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Abstract

We consider G-equivariant semilinear parabolic equations where G is a finite-dimensional possibly non-compact symmetry group. We treat periodic forcing of relative equilibria and resonant periodic forcing of relative periodic orbits as well as Hopf bifurcation from relative equilibria to relative periodic orbits using Lyapunov-Schmidt reduction. Our main interest are drift phenomena caused by resonance. In comparison to a center manifold approach Lyapunov-Schmidt reduction is technically easier. We discuss impacts of our results on spiral wave dynamics.

AMS subject classification. 35B32, 35K57, 57S20, 57S30
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1 Introduction

1.1 Spiral wave dynamics

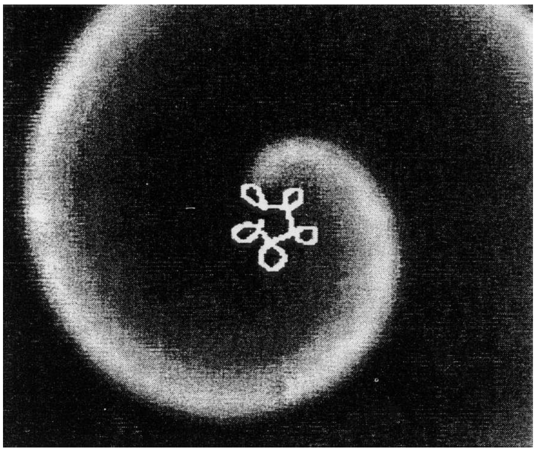

Figure 1: Meandering spiral wave in the Belousov Zhabotinsky reaction, from Steinbock et al. [28], with kind permission of Nature. The tip trajectory is overlaid with a white curve.

Relative equilibria and relative periodic solutions are ubiquitous in systems with continuous symmetry. Examples of relative equilibria and relative periodic solutions are spiral waves. Spiral waves have been observed in various chemical and biological systems, for example in the Belousov-Zhabotinsky reaction [5], [27], [35], and in catalysis on platinum surfaces [17].

The spiral tip of a rigidly rotating spiral wave moves on a circle. In mathematical terms rigidly rotating spiral waves are rotating waves. Rotating waves are stationary in a corotating frame and therefore examples of relative equilibria. Meandering spiral waves are modulated rotating waves, i.e., they are periodic in a corotating frame. In this case the spiral tip performs a quasiperiodic motion, which is called meandering, see Fig. 1.

Meandering spiral waves are generated by external periodic forcing of rigidly rotating spiral waves [17]. Let $\omega_{\rm ext}$ be the frequency of the external forcing and let $\mu_{\rm ext}$ be its amplitude. If the periodic forcing is resonant, i.e., if the rotation frequency $\omega^*_{\rm rot}$ of the rigidly rotating wave at $\mu_{\rm ext} = 0$ is a

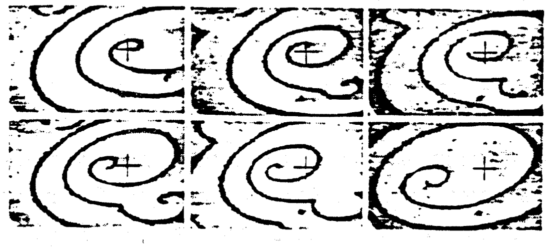

Figure 2: Drifting Spiral Waves in the CO-Oxidation on Pt(110), courtesy of [17]. The cross is always at the same position. So we see that the spiral wave drifts away from the cross.

multiple of the external frequency ω_{ext} of the system then a curve of drifting spiral waves in the $(\omega_{\text{ext}}, \mu_{\text{ext}})$-plane is observed, cf. [17], which separates modulated rotating wave states with inward petals and outward petals. This phenomenon is called resonance drift. Drifting spiral waves, see Fig. 2, are modulated travelling waves, i.e., they are periodic in a comoving frame. Both, meandering and drifting spiral waves are examples of relative periodic orbits.

In experiments also meandering spiral waves have been forced periodically [35]. Here invariant 3-tori are found and frequency locking between the period of the relative periodic orbits and the period of the external forcing occurs. Furthermore for certain external periods modulated travelling waves are generated. Experimentalists call this phenomenon generalized resonance drift [35].

Meandering spiral waves can also emanate from rigidly rotating spiral waves by a spontaneous bifurcation in autonomous systems, see [27], [32]. Barkley found in numerical simulations [3], see Fig. 3, that this transition is a Hopf bifurcation in the corotating frame. Hopf-bifurcation in autonomous systems leads to analogous drifting phenomena as periodic forcing of rigidly rotating waves.

The media in which spiral waves occur can be modelled by reaction-diffusion systems of the form

$$\frac{\partial u_i}{\partial t} = \delta_i \Delta u_i + f_i(u, t, \mu), \quad i = 1, \ldots, M. \tag{1.1}$$

Here $u = (u_1, \ldots, u_M)$ is a vector of concentrations of chemical species, the functions u_i, $i = 1, \ldots, M$, map the plane $\mathbb{R}^2$ to $\mathbb{R}$, the constants $\delta_i \geq 0$,

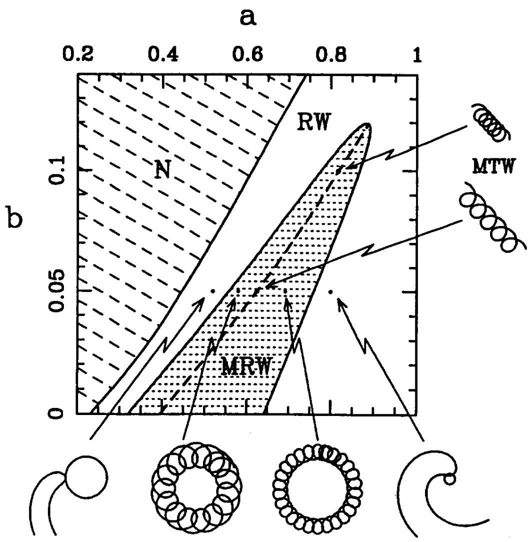

Figure 3: Phase diagram for the spiral wave dynamics depending on the parameters a, b; courtesy of Barkley [4]. Shown are regions containing N: no spiral waves, RW: stable rigidly rotating waves, MRW: modulated rotating waves, MTW: modulated travelling waves (dashed curve). Spiral tip paths illustrate states at 6 points. Small portions of spiral waves are shown for the two rotating wave cases.

$i = 1, \ldots, M$, are diffusion coefficients, $\mu \in \mathbb{R}^p$ is a parameter, and the functions f_i, $i = 1, \ldots, M$, are reaction-terms which are autonomous or time-periodic. Barkley [4] was the first to notice the importance of the Euclidean symmetry for spiral wave dynamics. The Euclidean group $\mathrm{E}(2) = \mathrm{O}(2) \ltimes \mathbb{R}^2$ of rotations, translations and reflections on the plane acts on the functions $u(x)$, $x \in \mathbb{R}^2$, via

$$(1.2) \qquad (\rho_{(R,a)}u)(x) = u(R^{-1}(x-a)), \quad \text{where} \quad R \in \mathrm{O}(2), a \in \mathbb{R}^2.$$

System (1.1) is equivariant with respect to the symmetry group $\mathrm{E}(2)$.

In this article we want to study the transition from rigidly rotating to meandering spiral waves on the infinitely extended plane $\mathbb{R}^2$. More generally the aim of the paper is to understand the transition from relative equilibria to relative periodic orbits in equivariant systems. Furthermore we want to explain the drift and resonance effects which we just described for general symmetry groups. We will discuss implications of our results on spiral wave dynamics in the plane and on the sphere (for simulations of spiral waves on the sphere see [36]). Further we want to apply our results to the evolution of scroll-waves in three-dimensional excitable media. Scroll waves have been studied numerically for example in [15], [19].

1.2 Related literature

In the thesis [33] the first results on bifurcations from rotating waves in systems with a non-compact, non-commutative symmetry group have been obtained. This paper is based on the dissertation [33]; but whereas in [33] we restricted attention to the symmetry group E(2) and applications in spiral wave dynamics in this article we treat arbitrary symmetry groups. As in [33] we study the transition from relative equilibria to relative periodic orbits using Lyapunov-Schmidt reduction.

Shortly after [33] was finished a whole bunch of papers on spiral wave dynamics and non-compact symmetry groups appeared:

Golubitsky et al. [10] used a formal center-bundle construction to derive ordinary differential equations describing bifurcations near ℓ-armed planar spiral waves of autonomous reaction-diffusion systems and derived new conditions for drifting. In [1] the drift of relative equilibria and periodic orbits along their group orbit is analyzed for general non-compact groups. Fiedler et al. [7] clarified the structure of the autonomous ordinary differential equations near relative equilibria with compact isotropy for general non-compact groups and gave conditions for drifting. In [22], [23] we presented a center-manifold reduction near relative equilibria and derived rigorously the ordinary differential equations on the center-manifold which were already guessed in [4] and formally derived in [10]. In [24] we extended these results to relative periodic orbits. In [8] normal forms near relative equilibria of non-compact group actions are computed. In [34] bifurcations from relative periodic orbits are treated.

Scheel [25], [26] proved the existence of rotating waves in unbounded domains.

The thesis [33] was inspired by work of Renardy on bifurcations from rotating waves [20]. Renardy also studied bifurcations from rotating waves of semilinear differential equations using Lyapunov-Schmidt reduction and applied his results to the Laser equations [21]. But his results for partial differential equations are restricted to compact symmetry groups.

1.3 Lyapunov-Schmidt-reduction versus center-manifold theory

To analyze bifurcations there are mainly two reduction methods: center-manifold reduction and Lyapunov-Schmidt reduction. Both have advantages and disadvantages. Here we will use Lyapunov-Schmidt reduction as tool for the analysis of bifurcations; for a center-manifold approach see [22], [23]. The advantage of Lyapunov-Schmidt reduction versus center-manifold theory is that we obtain C^∞-paths of relative periodic orbits if the nonlinearity in (1.1) is C^∞ whereas we only obtain a C^k-smooth center-manifold, $k < \infty$. Besides this we do not need the assumptions that the group action is isometric and that the group orbit of the relative equilibrium is an embedded manifold which are necessary for the center-manifold reduction. Finally the proofs are simpler since they do not rely upon the highly developed invariant manifold machinery. On the other hand the Lyapunov-Schmidt method is limited to relative equilibria and relative periodic orbits – we cannot handle more complicated dynamics. But for our purposes this is sufficient.

1.4 Organization of the paper

The paper is organized as follows.

First, in subsections 1.5 and 1.6 we study the functional-analytic framework of spiral wave dynamics and show some of the difficulties arising in the mathematical treatment of spiral waves. In subsection 1.7 we define an appropriate abstract setting which covers the reaction-diffusion system (1.1) modelling spiral wave dynamics. In this abstract setting we henceforth work. In section 2 we study periodic forcing of relative equilibria and relative periodic orbits. First, in subsection 2.1 we consider periodic forcing of relative equilibria and resonance drift. In subsection 2.3 we study the scaling of the drift velocity. As example we consider periodic forcing of rotating waves in E(2)-equivariant systems which lead to modulated rotating waves or, in the resonance case, to modulated travelling waves. This explains the experiments described in subsection 1.1. In subsection 2.4 we consider resonant periodic forcing of relative periodic orbits and discuss conditions for generalized resonance drift and the scaling of the drift velocity. The results apply to periodic forcing of meandering spiral waves as investigated experimentally by [35], see also subsection 1.1. In section 3 we discuss Hopf bifurcation from relative equilibria, resonances, scaling of drift velocity and effects of spatial isotropy of the relative equilibrium. As an example we study the Hopf bifurcation from multi-armed spiral waves. Section 4 is devoted to the proof of the main results.

1.5 Functional-analytic framework

To describe spiral wave dynamics we consider reaction-diffusion systems of the form (1.1) on a domain $\Omega \subset \mathbb{R}^3$ to $\mathbb{R}$, where Ω is a C^∞-manifold without boundary, for example $\mathbb{R}^2$, the unit sphere S^2 in $\mathbb{R}^3$ or $\mathbb{R}^3$ itself. The reaction-terms f_i, $i = 1, \dots, M$, are assumed to be C^k-smooth functions where $k \in \mathbb{N}$. The domain Ω is invariant under some subgroup G of the Euclidean group $\mathrm{E}(3)$ of motions in three-dimensional space consisting of rotations, reflections and translations. The group $\mathrm{E}(3) = \mathrm{O}(3) \ltimes \mathbb{R}^3$ acts on the functions $u(x)$, $x \in \mathbb{R}^3$, via (1.2), i.e.

$$(\rho_{(R,a)}u)(x) = u(R^{-1}(x-a)), \quad \text{where} \quad R \in \mathrm{O}(3), a \in \mathbb{R}^3.$$

System (1.1) is equivariant with respect to the group G. If $G = \mathrm{E}(2)$ is the Euclidean group of motions in the plane we write (ϕ, a) for (R_ϕ, a) where R_ϕ is a rotation with angle ϕ and $a \in \mathbb{R}^2$.

We consider (1.1) in the space of bounded uniformly continuous functions $X = BC_{\mathrm{unif}}(\Omega, \mathbb{R}^M)$ or in the space $X = L^2(\Omega, \mathbb{R}^M)$.

In $X = BC_{\mathrm{unif}}$ we get a time-evolution Φ_{t,t_0} of (1.1) on $Y = X$; if $X = L^2$ we obtain a time-evolution on $Y = X^\alpha$, $\alpha > 1/2$ without any growth conditions on f provided that $f(0,t,\mu) = 0$ for all t, μ and $\delta_i > 0$, $i = 1, \dots, M$. If $\delta_i = 0$ for some i we still obtain a semiflow on $X = H^2$ provided that $f(0,t,\mu) \equiv 0$.

Note that the group action is not smooth on the whole function space X. If the domain is $\Omega = \mathbb{R}^2$ and we choose $X = BC_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R}^M)$ then the $\mathrm{E}(2)$-action is even not strongly continuous because on the function $u(x_1, x_2) = \cos x_1$ the rotation acts discontinuously: For large radius r the term $|(\rho_{(\phi,0)}u)(x) - u(x)|$ can become equal to 2 even for arbitrarily small ϕ. We encounter the same problem if $\Omega = \mathbb{R}^3$. Since we want to have a strongly continuous group action on our base space X we consider the reaction-diffusion system (1.1) on a subspace of BC_{unif} which is invariant under the semiflow and where the group acts in a strongly continuous way:

We define $BC_{\mathrm{Eucl}}(\mathbb{R}^N, \mathbb{R}^M)$ as the subspace of $BC_{\mathrm{unif}}(\mathbb{R}^N, \mathbb{R}^M)$ on which $\mathrm{E}(N)$ acts continuously, $N = 2, 3$. The Laplacian is sectorial on $X = BC_{\mathrm{unif}}$

and on L^2, see [13]. We will now show that the Laplacian is also sectorial on $X = BC_{\mathrm{Eucl}}(\mathbb{R}^N, \mathbb{R}^M)$: let Y be any Banach space with a group G acting on it by a (not necessarily strongly continuous) representation ρ_g, $g \in G$. Let Y_0 be the subspace of Y on which G acts strongly continuously. If A is sectorial on Y and $A\rho_g = \rho_g A$ for all $g \in G$ then A is sectorial in Y_0: from $\rho_g \mathrm{e}^{-At} = \mathrm{e}^{-At}\rho_g$ we deduce that $(\mathrm{e}^{-At})_{t\geq 0}$ is a C^0-semigroup from Y_0 to Y_0; furthermore $\mathrm{e}^{-At}y$ is complex differentiable in t for $y \in Y$, $t > 0$, with derivative $A\mathrm{e}^{-At}y \in Y$. Since $\rho_g A\mathrm{e}^{-At} = A\mathrm{e}^{-At}\rho_g$ and therefore $A\mathrm{e}^{-At}Y_0 \subset Y_0$ we conclude that $(\mathrm{e}^{-At})_{t\geq 0}$ is an analytic semigroup on Y_0. Since $(\lambda - A)^{-1}u \in Y_0$ for $u \in Y_0$, $\lambda \in \mathbb{C}$, $\lambda \notin \mathrm{spec}_Y(A)$, the spectrum of A on Y_0 is contained in the spectrum of A on Y. Especially the Laplacian is sectorial on BC_{Eucl}, and its spectrum is contained in the spectrum of the Laplacian defined on BC_{unif}.

We also get a time-evolution of (1.1) in $BC_{\mathrm{Eucl}}(\mathbb{R}^N, \mathbb{R}^M)$ because we have $\rho_g \Phi_{t,t_0}(u) = \Phi_{t,t_0}(\rho_g u)$ and therefore Φ_{t,t_0} maps Y_0 into itself.

Now we have a C^0-group action on $X = BC_{\mathrm{Eucl}}$, but if $\Omega = \mathbb{R}^2, \mathbb{R}^3$ the semiflow does not smoothen the group-action even if all diffusion coefficients δ_i are positive. We demonstrate this for $\Omega = \mathbb{R}^2$ and for the heat equation where the nonlinearity f is zero.

We will show that on $\mathbb{R}^2$ the operator $\frac{\partial}{\partial\phi}$ is not bounded w.r.t. the Laplacian Δ and to the semiflow $(\mathrm{e}^{\Delta t})_{t\geq 0}$:

Remark 1.1 *The operator $\frac{\partial}{\partial\phi}$ is not bounded relatively to the Laplacian Δ or relatively to the semiflow $\mathrm{e}^{\Delta t}$, $t \geq 0$, on $BC_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R})$ and $BC_{\mathrm{Eucl}}(\mathbb{R}^2, \mathbb{R})$.*

Proof. The functions $w_{\ell,b}(x) := J_\ell(b|x|)\mathrm{e}^{\mathrm{i}\ell \arg(x)}$ where $b \geq 0$ and J_ℓ is the ℓ-th Bessel function of the first kind are elements of $BC_{\mathrm{Eucl}}(\mathbb{R}^2, \mathbb{R}) \subset BC_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R})$ and they are eigenfunctions of the Laplacian Δ and of the angle derivative $\frac{\partial}{\partial\phi}$:

$$\frac{\partial}{\partial\phi} w_{\ell,b} = \mathrm{i}\ell w_{\ell,b}, \quad \Delta w_{\ell,b} = -b^2 w_{\ell,b}.$$

Since $\mathrm{i}\ell(1+b^2)^{-1}$ and $\mathrm{i}\ell\mathrm{e}^{-b^2 t}$ are not bounded for arbitrary $b \in \mathbb{R}$, $\ell \in \mathbb{N}_0$, we conclude that $\frac{\partial}{\partial\phi}$ is not bounded relatively to Δ on $BC_{\mathrm{Eucl}}(\mathbb{R}^2, \mathbb{R})$, $BC_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R})$ and that $\frac{\partial}{\partial\phi}\mathrm{e}^{\Delta t}$ is not a bounded operator on $BC_{\mathrm{Eucl}}(\mathbb{R}^2, \mathbb{R})$, $BC_{\mathrm{unif}}(\mathbb{R}^2, \mathbb{R})$ for $t > 0$. ∎

Remark 1.2 *Also on $L^2(\mathbb{R}^2, \mathbb{R})$ the angle-derivative $\frac{\partial}{\partial\phi}$ is not bounded relatively to Δ or $\mathrm{e}^{\Delta t}$, $t \geq 0$.*

Proof. By direct computation we see that $\mathcal{F}(\frac{\partial}{\partial\phi}u) = \frac{\partial}{\partial\phi}\mathcal{F}(u)$. Here $\mathcal{F}(u)$ denotes the Fourier transform of u. From this formula and from $\mathcal{F}(\Delta u)(x) = -|x|^2\mathcal{F}(u)(x)$ we deduce that $\frac{\partial}{\partial\phi}$ is not bounded with respect to Δ. Furthermore the operator $\frac{\partial}{\partial\phi}$ is not bounded relatively to $\mathrm{e}^{\Delta t}$ in $L^2(\mathbb{R}^2,\mathbb{R})$ since $(\mathcal{F}(\frac{\partial}{\partial\phi}\mathrm{e}^{\Delta t}u))(x) = \frac{\partial}{\partial\phi}\mathrm{e}^{-|x|^2 t}(\mathcal{F}(u))(x)$ is not defined for all $u \in L^2(\mathbb{R}^2,\mathbb{R})$. ∎

Therefore we cannot simply change coordinates into a corotating frame to deal with the meandering transition.

1.6 Representations of $\mathrm{E}(N)$

The function spaces $Y = BC_{\mathrm{Eucl}}(\mathbb{R}^N,\mathbb{R}), L^2(\mathbb{R}^N,\mathbb{R})$, $N = 2,3$, do not contain finite-dimensional subspaces which are $\mathrm{E}(N)$-invariant and in which the $\mathrm{E}(N)$-action is non-trivial. Again we will demonstrate this in the case $\Omega = \mathbb{R}^2$, $G = \mathrm{E}(2)$:

Lemma 1.3 *Let the action of* $\mathrm{E}(2)$ *on the spaces* $X = BC_{\mathrm{Eucl}}(\mathbb{R}^2,\mathbb{R})$, $X = L^2(\mathbb{R}^2,\mathbb{R})$ *be given by (1.2). Then the function spaces* BC_{Eucl}, L^2 *do not contain finite-dimensional* $\mathrm{E}(2)$*-invariant subspaces with nontrivial* $\mathrm{E}(2)$*-action.*

In Greenleaf [12] a general theory on the action of topological groups on function spaces is developed.

If we allow polynomial growth in our function space then the polynomials of degree $\leq j$ are finite-dimensional representations of $\mathrm{E}(2)$.

Proof of Lemma 1.3. Let $V_j = \mathrm{span}(e_1,\ldots,e_j)$ be a j-dimensional representation of $\mathrm{E}(2)$ in BC_{unif} or L^2. Then the translations act as a C^0-group of isometries on V_j since they act in such a way on BC_{unif}, L^2. Since V_j is finite-dimensional, we know that $\rho_{(0,(a_1,a_2))}e_i = \sum_{i=1}^{j}(\mathrm{e}^{\eta_1 a_1+\eta_2 a_2})_{ij}e_j$ where $\eta_1 = \frac{\partial}{\partial x_1}|_{V_j}$, $\eta_2 = \frac{\partial}{\partial x_2}|_{V_j}$ are (j,j)-matrices. Since $\rho_{(0,a)}$ is an isometry we conclude that $\mathrm{Re\,spec}(\eta_1) = \mathrm{Re\,spec}(\eta_2) = 0$ and that η_1,η_2 do not contain Jordan blocks. After simultaneous diagonalization of η_1, η_2 (note that $[\eta_1,\eta_2] = 0$) we see that the eigenfunctions of η_1, η_2 are of the form e^{ibx}, $b, x \in \mathbb{R}^2$. These functions are not elements of $X = L^2(\mathbb{R}^2,\mathbb{R})$. So the proof is finished for the function space L^2. If we choose $b = 0$ we obtain an $\mathrm{E}(2)$-invariant subspace of $X = BC_{\mathrm{unif}}(\mathbb{R}^2,\mathbb{R})$ which consists of all constant functions. The $\mathrm{E}(2)$-action on this space is trivial. The action of the rotation is not continuous on the functions e^{ibx}, $b \neq 0$, with respect to the norm

$\| \cdot \|_{BC_{\text{unif}}(\mathbb{R}^2,\mathbb{R})}$. Therefore the functions e^{ibx} do not span a finite-dimensional E(2)-invariant subspace of $BC_{\text{Eucl}}(\mathbb{R}^2,\mathbb{R})$ for $b \neq 0$. ∎

Of course, the same considerations apply for $x \in \mathbb{R}^3$, $G = \mathrm{E}(3)$ instead of $x \in \mathbb{R}^2$, $G = \mathrm{E}(2)$.

Especially for an E(2)-invariant steady state the eigenspace to each eigenvalue is E(2)-invariant and therefore infinite-dimensional. This makes the study of bifurcations from E(2)-invariant equilibria for an abstract equivariant parabolic equation very difficult. We will not attack this problem and rather study bifurcations from relative equilibria where these difficulties do not occur. Bifurcations from homogeneous steady states of reaction diffusion equations have been studied by Scheel [25], [26] using spatial dynamics.

1.7 Abstract Setting

In this paper we study semilinear parabolic equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -Au + f(u, \omega_{\text{ext}} t, \mu) \tag{1.3}$$

on some Banach space X which are equivariant under a m-dimensional Lie group G which may be non-compact. We assume that A is sectorial (for a definition see [13]) and that f is C^k-smooth from $Y \times \mathbb{R} \times \mathbb{R}^p$ to X. Here $k \in \mathbb{N}$ or $k = \infty$, $\mu \in \mathbb{R}^p$ and $Y = X^\alpha$ for $0 \leq \alpha < 1$.

By [13] there exists a time-evolution $\Phi_{t,t_0}(\cdot;\mu)$ of (1.3) on Y, and $\Phi_{t,t_0}(u;\mu)$ is C^k-smooth in u, μ for $t \geq t_0$ and in u, μ, t, t_0 for $t > t_0$. We assume that the group G acts on Y by the linear strongly continuous representation $\rho_g \in \mathcal{L}(Y)$, $g \in G$ and that (1.3) is G-equivariant, i.e.,

$$\forall\, g \in G \quad \rho_g A = A \rho_g, \quad f(\rho_g u, t, \mu) = \rho_g f(u, t, \mu)$$

This implies that $\rho_g \Phi_{t,t_0}(\cdot;\mu) = \Phi_{t,t_0}(\rho_g \cdot;\mu)$ for all $g \in G$.

Assume that f in (1.3) is time-independent. Then a group orbit Gu^* is called a *relative equilibrium* of (1.3) if $\Phi_t(u^*) = \rho_{\exp(\xi^* t)} u^*$ for some $\xi^* \in \mathrm{alg}(G)$. Here $\mathrm{alg}(G)$ denotes the Lie algebra of G. Sometimes we denote u^* itself as relative equilibrium.

A point u^* lies on a *relative periodic orbit*

$$\mathcal{O}^* = \{\rho_g \Phi_{t,0}(u^*) \mid g \in G, t \in \mathbb{R}\}$$

if $\Phi_{T^*,0}(u^*) = \rho_{g^*} u^*$ for some $T^* > 0$, $g^* \in G$. In this case we suppose that $f(u, \omega_{\text{ext}} t, \mu)$ is independent of time or time-periodic with frequency

$\omega_{\rm ext} = 2\pi j/T^*$, $j \in \mathbb{N}$. Sometimes we sloppily denote u^* itself as relative periodic orbit. We call T^* the relative period of the relative periodic orbit.

The aim of this article is to study transitions from relative equilibria to relative periodic orbits of (1.3).

2 Periodically forced G-equivariant systems

This section deals with the effects of periodic forcing on relative equilibria and relative periodic orbits. In particular, we will investigate drift phenomena caused by resonant periodic forcing. We will apply our results to spiral wave dynamics. This helps to understand the experiments mentioned in the introduction. Proofs of the main theorems are postponed to section 4.

In this section we assume that the nonlinearity f of (1.3) is of the form

$$f(u,t,\mu) = \hat{f}(u,\hat{\mu}) + \mu_{\rm ext} f_{\rm ext}(u, \omega_{\rm ext} t, \mu).$$

Here $f_{\rm ext}(u,\tau,\mu)$ is 2π-periodic in τ; $\omega_{\rm ext}$ is the frequency of the periodic forcing, $T_{\rm ext} = \frac{2\pi}{\omega_{\rm ext}}$ is its period, $\mu_{\rm ext}$ is its amplitude and we decompose $\mu = (\mu_{\rm ext}, \hat{\mu})$, where $\mu_{\rm ext} \in \mathbb{R}$, $\hat{\mu} \in \mathbb{R}^{p-1}$. So we consider the periodically forced differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -Au + \hat{f}(u,\hat{\mu}) + \mu_{\rm ext} f_{\rm ext}(u, \omega_{\rm ext} t, \mu). \tag{2.1}$$

A typical example of the abstract semilinear differential equation (2.1) is a periodically forced reaction-diffusion system on the domain $\Omega \subset \mathbb{R}^N$, $N = 2, 3$, cf. (1.1):

$$\frac{\partial u_i}{\partial t} = \delta_i \Delta u_i + \hat{f}_i(u,\hat{\mu}) + \mu_{\rm ext} f_{{\rm ext},i}(u, \omega_{\rm ext} t, \mu), \quad i = 1, \ldots, M. \tag{2.2}$$

2.1 Periodic forcing of relative equilibria

This subsection deals with effects of periodic forcing on relative equilibria. First we state two general theorems, then we study examples in spiral wave dynamics.

Consider system (2.1) without periodic forcing, i.e., at $\mu_{\rm ext} = 0$. Assume that u^* is a relative equilibrium of the unforced system for the parameter $\hat{\mu} = \hat{\mu}^* = 0$. Then u^* satisfies

$$\Phi_t(u^*) = \rho_{e^{t\xi^*}} u^*$$

for some $\xi^* \in \mathrm{alg}(G)$. Since $\Phi_t(\cdot)$ is equivariant and C^k-smooth in t for $t > 0$ we conclude that $\mathrm{e}^{t\xi^*}u^*$ is C^k-smooth in t for all $t \in \mathbb{R}$.

We will write ξu for $\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\mathrm{e}^{t\xi}}u|_{t=0}$. Furthermore denote by

$$\mathrm{Ad}_g\,\xi := g\xi g^{-1} = \frac{\mathrm{d}}{\mathrm{d}t}(g\exp(\xi t)g^{-1})\Big|_{t=0} \in \mathrm{alg}(G)$$

the adjoint action of G on $\mathrm{alg}(G)$ and by

$$K = \{g \in G \mid \rho_g u^* = u^*\}$$

the isotropy group of u^*. We assume that K is compact. Let G^0 denote the identity component of G. We have $\xi^* \in \mathrm{alg}(N(K))$ where $N(K)$ is the normalizer of the isotropy group K of u_* because for $g \in K$, $t \in \mathbb{R}$,

$$\rho_g\rho_{\exp(t\xi^*)}u^* = \rho_g\Phi_t(u^*) = \Phi_t(\rho_g u^*) = \Phi_t(u^*) = \rho_{\exp(t\xi^*)}u^*$$

and therefore $g\exp(t\xi^*) \in \exp(t\xi^*)K$. Similarly the pull-back element g^* of a relative periodic orbit $u^* = \rho_{g^*}^{-1}\Phi_{T_{\mathrm{ext}},0}(u^*)$ lies in the normalizer of the isotropy K of u^*. Actually for a relative equilibrium the drift velocity ξ^* lies in the Lie algebra of the centralizer $Z(K)$ of K, which follows from the formula $N(K)^0 = K^0 Z(K)^0$, see [9].

Since by periodic forcing isotropy is not changed we assume without loss of generality in the whole section that $K = \{\mathrm{id}\}$. Otherwise we change the space Y to the fixed point space $\mathrm{Fix}(K)$ of K and the symmetry group G to $N(K)/K$.

Let u^* be a relative equilibrium, i.e. $-Au^* + \hat{f}(u^*) = \xi^* u^*$, and let

$$L^* = -A + \mathrm{D}_u\hat{f}(u^*) - \xi^*$$

be the linearization at the relative equilibrium in the comoving frame. We compute that for $\xi \in \mathrm{alg}(G)$

$$\begin{aligned} L^*\xi u^* &= (-A + \mathrm{D}_u\hat{f}(u^*) - \xi^*)\xi u^* \\ &= -\xi A u^* + \mathrm{D}_u\hat{f}(u^*)\xi u^* - \xi^*\xi u^* \\ &= \xi(-A + \hat{f}(u^*)) - \xi^*\xi u^* \\ &= (\xi\xi^* - \xi^*\xi)u^* \\ &= [\xi, \xi^*]u^*. \end{aligned} \tag{2.3}$$

Here $[\cdot,\cdot]$ denotes the commutator and we used that $g\hat{f}(u) = \hat{f}(gu)$ and therefore $\mathrm{D}_u\hat{f}(u^*)\xi = \xi\hat{f}(u)$. From (2.3) we see that L^* maps $T_{u^*}Gu^* = \mathrm{alg}(G)u^*$ into itself.

Example 2.1 Let u^* be a rotating wave of the unforced system (2.1), e.g a rigidly rotating spiral wave of the reaction-diffusion system (2.2) on $\Omega = \mathbb{R}^2$ at $\mu_{\mathrm{ext}} = 0$. Then the symmetry group is $G = \mathrm{E}(2)$. We write $g = (\phi, a) \in \mathrm{SO}(2) \ltimes \mathbb{R}^2 = \mathrm{SE}(2)$. Let ξ_1 denote the generator of the rotation and ξ_2, ξ_3 denote the generators of the translation. Then $\xi^* = \omega^*_{\mathrm{rot}}\xi_1$ where ω^*_{rot} is the rotation frequency of the spiral, and we compute

$$L^*\xi_1 u^* = 0, \quad L^*(\xi_2 + \mathrm{i}\xi_3)u^* = \omega^*_{\mathrm{rot}}[\xi_2 + \mathrm{i}\xi_3, \xi_1]u^* = \mathrm{i}\omega^*_{\mathrm{rot}}(\xi_2 + \mathrm{i}\xi_3)u^*.$$

Therefore the linearization L^* of the rotating wave in the rotating frame has always eigenvalues on the imaginary axis.

For a relative periodic orbit $u^* = \rho^{-1}_{g^*}\Phi_{T^*,0}(u^*)$ we get

$$\rho^{-1}_{g^*}\mathrm{D}\Phi_{T^*,0}(u^*)\xi u^* = (\mathrm{Ad}^{-1}_{g^*}\,\xi)u^*, \quad \xi \in \mathrm{alg}(G).$$

If u^* is a relative equilibrium then the linearization of the time-T-map in the comoving frame ξ_* is given by

$$\mathrm{e}^{L^*T} = \rho^{-1}_{g^*}\mathrm{D}\Phi_T(u^*)$$

where $g^* = \mathrm{e}^{T\xi^*}$.

For the groups relevant in applications (compact and Euclidean groups) the eigenvalues of the linear maps $[\xi, \cdot]$ ($\xi \in \mathrm{alg}(G)$) on $\mathrm{alg}(G)$ are purely imaginary and similarly the spectrum of the maps Ad_g, $g \in G$, on $\mathrm{alg}(G)$ lies on the unit circle. We will restrict our attention to these groups in this article. So we make the overall hypothesis

Overall Hypothesis *The spectra of the linear maps* Ad_g, $g \in G$, *are subsets of the unit circle* $\{\lambda \in \mathbb{C} \;\; |\lambda| = 1\}$.

Therefore in the case of continuous symmetry where $\mathrm{alg}(G)$ is nontrivial the linearization L^* at a relative equilibrium always has eigenvalues on the imaginary axis and similarly the linearization $\rho^{-1}_{g^*}\mathrm{D}\Phi_T(u^*)$ of a relative periodic orbit $u^* = \rho^{-1}_{g^*}\Phi_T(u^*)$ of (2.1) has always center-eigenvalues on the unit circle.

If u^* is a relative equilibrium fix some $T > 0$. In the case of a relative periodic orbit take $T = T^*$. We need the following assumption on the spectrum:

Hypothesis (S) *The set* $\{\lambda \in \mathbb{C};\ |\lambda| \geq 1\}$ *is a spectral set for the spectrum* $\mathrm{spec}(B^*)$ *of the operator*

$$B^* := \rho^{-1}_{g^*}\mathrm{D}\Phi_T(u^*) \in \mathcal{L}(Y) \tag{2.4}$$

*(called center-unstable spectral set) with associated spectral projection $P \in \mathcal{L}(Y)$ and the corresponding generalized eigenspace $E_{\text{cu}} := \mathcal{R}(P)$ (the center-unstable eigenspace) is finite-dimensional.*Frequently we employ the following notion:

Definition 2.2 *We say that a relative periodic orbit or a relative equilibrium u^* of (2.1) is* non-critical *if the operator B^* from (2.4) satisfies Hypothesis (S) and if the center-eigenspace*

$$E_{\text{c}} = T_{u^*}Gu^* + \text{span}(\partial_t \Phi_t(u^*)|_{t=0})$$

only consists of eigenvectors which are forced by G-symmetry or time-shift symmetry (in the case of relative periodic orbits of autonomous systems).

Denote the dual space of Y by $Y^\star$ and let $m = \dim(G)$. Choose $l_i \in Y^\star$, $i = 1, \ldots, m$, such that the equations $l_i(u - u^*) = 0$, , $i = 1, \ldots, m$, define a section transverse to the group orbit Gu^* of the relative equilibrium at u^*. If u^* is non-critical we can choose the functionals l_i as left center-eigenvectors of L^*.

The following theorem essentially states that external periodic forcing leads to a transition from relative equilibria to relative periodic orbits.

Theorem 2.3 *Let $u^* = \rho_{\mathrm{e}^{-t\xi^*}} \Phi_t(u^*)$ be a relative equilibrium of the unforced system (2.1), i.e., for the parameter $\mu = 0$. Compute $B^* = \mathrm{e}^{T^*_{\text{ext}} L^*}$ as in (2.4) and assume that u^* satisfies assumption (S).*

If the generalized eigenspace of B^* to the eigenvalue 1 lies in $\text{alg}(G)u^*$ then for each small amplitude μ_{ext} of the periodic forcing, each frequency $\omega_{\text{ext}} \approx \omega^*_{\text{ext}}$ of the forcing and each small $\hat{\mu}$ there is exactly one relative periodic orbit $u = u(\omega_{\text{ext}}, \mu)$, of (2.1) satisfying

$$(2.5) \qquad u = \rho_g^{-1} \Phi_{T_{\text{ext}},0}(u, \mu) \quad \textit{and} \quad l_i(u - u^*) = 0, \quad i = 1, \ldots, m,$$

for some $g = g(\omega_{\text{ext}}, \mu)$. Furthermore $\rho_g u(\omega_{\text{ext}}, \mu)$ is C^k in $g \in G$, ω_{ext} and μ, $g(\omega_{\text{ext}}, \mu)$ is C^k in $(\omega_{\text{ext}}, \mu)$ and $u(\omega_{\text{ext}}, 0) = u^*$, $g(\omega_{\text{ext}}, 0) = g^*$.

Often we need not use the full symmetry G of (3.1) to prove Theorem 2.3. If L^* does not have eigenvalues $\mathrm{i}j\omega^*_{\text{ext}}$, $j \in \mathbb{Z}$, forced by symmetry then the symmetry group is discrete and we need not take it into account to prove the theorem. If $[\cdot, \xi^*]$ has eigenvalues in $\mathrm{i}\omega^*_{\text{ext}}\mathbb{Z}$, then the corresponding (generalized) eigenvectors form a Lie-subalgebra of $\mathrm{alg}(G)$ as can be seen from the Jacobi-identity.

We call the Lie group generated by the generalized eigenvectors of $[\cdot, \xi^*]$ to the spectral set $\mathrm{i}\omega^*_{\mathrm{ext}}\mathbb{Z}$ the *minimal symmetry group* for the forcing frequency ω^*_{ext} that we consider.

2.2 Resonance drift

Now we deal with the effects of resonant periodic forcing. We need the following notion:

Definition 2.4 *Let $g \in G$. If $g^n = \exp(\xi n)$ for some $\xi \in \mathrm{alg}(G)$ with $\mathrm{Ad}_g \xi = \xi$ and $n \in \mathbb{N}$ then we call ξ* average velocity *of g.*

There may be many average velocities for each group element g; for example if $G = \mathrm{SO}(2)$ then for $g^* = \phi^*$ the set $\{\xi^* = \phi^* + j2\pi \mid j \in \mathbb{Z}\}$ consists of average velocities for g^*. If $u = \rho_g^{-1}\Phi_{T,0}(u)$ is a relative periodic orbit of (2.1) and ξ is an average velocity of g then we call ξ/T average velocity of the relative periodic orbit.

Definition 2.5 *If $\exp(\cdot)$ is not locally surjective near $\xi^* \in \mathrm{alg}(G)$ then there are elements $g \in G$ close to $\exp(\xi^*)$ which have (if any) only average velocities ξ which are far away from ξ^*. We call this phenomenon* resonance drift.

Similarly, let u^* be a non-critical relative equilibrium of the unperturbed system (2.1) which travels with velocity ξ^*. If the period of the external forcing T^*_{ext} is such that $\exp(\cdot)$ is not locally surjective near $\xi = \xi_* T^*_{\mathrm{ext}}$ then it may happen that relative periodic orbits of (2.1) which are generated by external periodic forcing, see Theorem 2.3, drift with an average velocity completely different to the drift velocity ξ^* of the relative equilibrium at $\mu_{\mathrm{ext}} = 0$. We also call this effect *resonance drift*.

For resonance drift to occur it is necessary that the periodic forcing is resonant which means that the linearization L^* of the relative equilibrium in the comoving frame has a symmetry eigenvalue in $\mathrm{i}\omega^*_{\mathrm{ext}}\mathbb{Z} \setminus \{0\}$. Otherwise $\exp()$ would be surjective near $T^*_{\mathrm{ext}}\xi^*$ and the relative periodic orbits $u(\mu)$ generated by periodic forcing would drift with velocity $\xi(\mu) \approx \xi^*$.

As we mentioned in the introduction even a transition from compact to noncompact drift may take place. We will deal with this in the following example:

Example 2.6 Consider Example 2.1 again: Let the symmetry group be $G = \mathrm{E}(2)$, write $g = (\phi, a) \in \mathrm{SO}(2) \ltimes \mathbb{R}^2 = \mathrm{SE}(2)$ and let u^* be a non-critical rotating wave $u^* = \rho_{(-\omega_{\mathrm{rot}}^* t, 0)} \Phi_t(u^*)$ of the unforced system (2.1), $\mu_{\mathrm{ext}} = 0$. For example u^* could be a rigidly rotating spiral wave of the reaction-diffusion system (2.2) on $\Omega = \mathbb{R}^2$. By Theorem 2.3 for each small forcing amplitude $\mu_{\mathrm{ext}} \approx 0$ and each forcing frequency ω_{ext} there is a relative periodic orbit $u(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}}) \approx u^*$.

If $\omega_{\mathrm{rot}}^*/\omega_{\mathrm{ext}}^* \notin \mathbb{Z}$ then the forcing is non-resonant and the relative periodic orbits $u(\mu_{\mathrm{ext}}, \omega_{\mathrm{ext}})$ with $\omega_{\mathrm{ext}} \approx \omega_{\mathrm{ext}}^*$ are modulated rotating waves of (2.1) (called meandering spiral waves in the example (2.2)).

If $\omega_{\mathrm{rot}}^*/\omega_{\mathrm{ext}}^* = j \in \mathbb{Z}$ then $\mathrm{D}\exp(2\pi\xi^*/\omega_{\mathrm{ext}}^*)$ has rank defect 2. We talk of a $j : 1$-resonance. In this case modulated travelling waves (called drifting spiral waves of (2.2)) are generated as the following proposition shows:

Proposition 2.7 *If a rotating wave of an* $\mathrm{E}(2)$*-equivariant system (2.1) is subject to* $j : 1$*-resonant periodic forcing then there is a* C^k*-smooth path* $u(\mu_{\mathrm{ext}})$, $a(\mu_{\mathrm{ext}})$, $\omega_{\mathrm{ext}}(\mu_{\mathrm{ext}})$, *of modulated travelling waves satisfying*

$$\Phi_{2\pi/\omega_{\mathrm{ext}}(\mu_{\mathrm{ext}})}(u(\mu_{\mathrm{ext}})) = \rho_{(0, a(\mu_{\mathrm{ext}}))} u(\mu_{\mathrm{ext}})$$

such that $u(0) = u^*$, $a(0) = 0$, $\omega_{\mathrm{ext}}(0) = \omega_{\mathrm{ext}}^*$.

Proof. By Theorem 2.3 we get a surface $u(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}})$ of relative periodic orbits satisfying (2.5) where $g(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}}) = (\phi(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}}), a(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}}))$. To obtain modulated travelling waves we need to solve the equation

$$\phi(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}}) = 0 \mod 2\pi.$$

We have $\partial_{\omega_{\mathrm{ext}}} \phi(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}})|_{(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}}) = (\omega_{\mathrm{ext}}^*, 0)} \neq 0$. This can be seen as follows: Let ξ_1 be the generator of the rotation, and ξ_2, ξ_3 be the generators of the translation. Computing the derivative w.r.t. ω_{ext} of (2.5) in $(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}}) = (\omega_{\mathrm{ext}}^*, 0)$ we get

$$\begin{aligned} -2\pi\omega_{\mathrm{rot}}^*/(\omega_{\mathrm{ext}}^*)^2 \xi_1 u^* + (\mathrm{D}\Phi_{T_{\mathrm{ext}}^*, 0}(u^*) - 1)\partial_{\omega_{\mathrm{ext}}} u(\omega_{\mathrm{ext}}^*, 0) \\ = (\partial_{\omega_{\mathrm{ext}}} \phi(\omega_{\mathrm{ext}}^*, 0)\xi_1 + \partial_{\omega_{\mathrm{ext}}} a_1(\omega_{\mathrm{ext}}^*, 0)\xi_2 + \partial_{\omega_{\mathrm{ext}}} a_2(\omega_{\mathrm{ext}}^*, 0)\xi_3)u^*. \end{aligned} \tag{2.6}$$

Here we used that

$$\partial_{\omega_{\mathrm{ext}}} \Phi_{2\pi/\omega_{\mathrm{ext}}^*}(u^*) = -2\pi/{\omega_{\mathrm{ext}}^*}^2 \partial_T \Phi_{2\pi/\omega_{\mathrm{ext}}^*}(u^*) = -2\pi\omega_{\mathrm{rot}}^*/\omega_{\mathrm{ext}}^2 \xi_1 u^*.$$

If we choose the l_i in (2.5) as left center-eigenvectors of L^* then

$$l_i((\mathrm{D}\Phi_{T^*_{\mathrm{ext}},0}(u^*)-1)\partial_{\omega_{\mathrm{ext}}}u(\omega^*_{\mathrm{ext}},0))=0,\quad i=1,2,3.$$

Applying the functionals l_i, $i=1,2,3$, onto (2.6) we conclude that

$$\partial_{\omega_{\mathrm{ext}}}\phi(\omega^*_{\mathrm{ext}},0)=-2\pi\omega^*_{\mathrm{rot}}/(\omega^*_{\mathrm{ext}})^2\neq 0.$$

Hence we can apply the implicit function theorem to get a smooth path $\mu_{\mathrm{ext}}(\omega_{\mathrm{ext}})$ parametrizing modulated travelling waves. ∎

A transition from rotating waves to modulated travelling waves has been observed in experiments [17] in the case of 1 : 1-resonance and 2 : 1-resonance.

Ashwin and Melbourne [2] talk of drift bifurcation of relative equilibria if a rotating wave of an E(2)-equivariant system becomes a travelling wave in the limit $\omega_{\mathrm{rot}}\to 0$. So their drift bifurcation and our resonance drift are related. But in our case the resonance drift is enforced by periodic forcing.

Example 2.8 Consider the reaction-diffusion system (2.2) on the sphere $\Omega=S^2$. Then the symmetry group is $G=\mathrm{O}(3)$. We will show that a wave u^* rotating around the x_3-axis starts meandering around some vector in the (x_1,x_2)-plane if it is subject to resonant periodic forcing.

Let ξ_i denote the generators of the rotation around the unit vectors $\mathbf{e}_i\in\mathbb{R}^3$, $i=1,2,3$, and write $g\in\mathrm{SO}(3)$ as $g=\exp(\sum_{i=1}^3\phi_i\xi_i)$. Let $u^*=\rho_{\exp(-\xi^*t)}\Phi_t(u^*)$ be a non-critical wave of the unforced system (2.2), $\mu_{\mathrm{ext}}=0$, rotating around the x_1-axis, i.e., $\xi^*=\omega^*_{\mathrm{rot}}\xi_1$. As in (2.3) we compute

$$L^*(\xi_2+\mathrm{i}\xi_3)u^*=\mathrm{i}\omega^*_{\mathrm{rot}}(\xi_2+\mathrm{i}\xi_3)u^*.$$

If we switch on resonant periodic forcing with $\omega^*_{\mathrm{ext}}=\omega^*_{\mathrm{rot}}/j$, $j\in\mathbb{Z}$, then there is a smooth path $u(\mu_{\mathrm{ext}})$, $\omega_{\mathrm{ext}}(\mu_{\mathrm{ext}})$ of waves meandering around some vector in the (x_2,x_3)-plane:

$$\Phi_{T_{\mathrm{ext}}(\mu_{\mathrm{ext}}),0}(u(\mu_{\mathrm{ext}}))=\rho_{\exp(\phi_2(\mu_{\mathrm{ext}})\xi_2+\phi_3(\mu_{\mathrm{ext}})\xi_3)}u(\mu_{\mathrm{ext}})$$

where $\phi_2(0)=0,\phi_3(0)=0$, $\omega_{\mathrm{ext}}(0)=\omega^*_{\mathrm{ext}}$, $u(0)=u^*$. This can be seen as in Example 2.6.

For numerical simulations of rotating waves on the sphere S^2 see [36].

In the last two examples of resonant forcing the relative equilibria were always rotating waves. But also for nonperiodic relative equilibria resonance drift occurs:

Example 2.9 Consider the reaction-diffusion system (2.2) in three space $\Omega = \mathbb{R}^3$. Then the symmetry group is the Euclidean group E(3).

Let u^* be a twisted scroll ring of the unforced system (2.2). Such a wave consists of a circular filament in the (x_2, x_3)-plane along which vertical spiral waves are located and an additional infinitely extended vertical filament [19]. It is a relative equilibrium which translates along its vertical filament and simultaneously rotates around it.

Because of the vertical filament only translations $a \in \mathbb{R}^3$ and rotations around the x_3-axis act continuously on u_* in the space BC_{unif}. So the effective symmetry group is in this case $G = \text{E}(2) \times \mathbb{R}$. cf. [24]. We write $g = (\phi, a)$ for the elements of $\text{E}(2) \times \mathbb{R}$ where ϕ is the rotation angle around the x_1-axis and $a \in \mathbb{R}^3$ is a translation vector.

The time-evolution of the twisted scroll ring is given by $\Phi_t(u^*) = \rho_{\exp(\xi^* t)} u^*$ where $\xi^* = (\omega^*_{\text{rot}}, v^* \mathbf{e}_1)$.

If the twisted scroll ring is forced periodically with frequency ω_{ext} it will typically start meandering in the (x_2, x_3)-plane:

$$\Phi_{T_{\text{ext}},0}(u(\mu_{\text{ext}})) = \rho_{(\phi(\mu_{\text{ext}}), a(\mu_{\text{ext}}))} u(\mu_{\text{ext}}), \quad a(\mu_{\text{ext}}) = v(\mu_{\text{ext}}) T_{\text{ext}} \mathbf{e}_1.$$

But by resonant periodic forcing, i.e. if $\omega^*_{\text{rot}} / \omega^*_{\text{ext}} \in \mathbb{Z}$, we can achieve that the scroll ring drifts away in another direction than the x_1-axis as the following proposition shows:

Proposition 2.10 *If the twisted scroll ring of (2.2) is noncritical and forced periodically such that $\omega^*_{\text{rot}} / \omega^*_{\text{ext}} \in \mathbb{Z}$ then there is a C^k-smooth path $u(\mu_{\text{ext}})$, $\omega_{\text{ext}}(\mu_{\text{ext}})$ of relative periodic orbits satisfying*

$$\Phi_{2\pi/\omega_{\text{ext}}(\mu_{\text{ext}}),0}(u(\mu_{\text{ext}})) = \rho_{(0, a(\mu_{\text{ext}}))} u(\mu_{\text{ext}}), \quad a(\mu_{\text{ext}}) \in \mathbb{R}^3.$$

The direction of the drift $a(\mu_{\text{ext}})$ of the periodically forced twisted scroll rings in the above proposition will typically not point in x_1-direction. The proof of the proposition is similar as the proof of Proposition 2.7.

Note again that to the isotropy K of the relative equilibria not all kinds of noncompact drift are possible. As mentioned before the drifts $g(\omega_{\text{ext}}, \mu)$ of the emanating relative periodic orbits have to lie in $N(K)$. Remember that we have chosen $G = N(K)/K$ in the whole section. In a second step we have to interpret our results on periodic forcing for the original group G. In a system with E(2)-symmetry for instance we see that a rotating wave with spatial symmetry K can not start drifting under the influence of the periodic forcing if K contains a non-trivial rotation $(\phi, 0)$. In this case $N(K) = \mathrm{SO}(2)$, see [7]. Similarly if $G = \mathrm{E}(2)$ and K only consists of one reflection then the relative equilibrium u^* can not rotate. Hence it is a travelling wave in general. A relative equilibrium in an E(2)-equivariant system with $K \supset D_n$, $n > 1$, even has to be stationary.

We can generalize Propositions 2.7, 2.10 as follows:

Let $g = \tilde{g}(\chi)$, $\chi \in \mathbb{R}^n$, $|\chi| \leq 1$, be a smooth n-dimensional hyper-surface in G such that $g(0) = g^* = \exp(T^*_{\mathrm{ext}}\xi^*)$. Let $\{\xi_i \mid i = 1, \dots m\}$, $m = \dim(G)$, denote a basis of $\mathrm{alg}(G)$. Write

$$\tilde{g}(\chi) = \exp(\tilde{\zeta}(\chi))g^*, \quad \tilde{\zeta}(\chi) = \sum_{i=1}^{\dim(G)} \tilde{\zeta}_i(\chi)\xi_i, \tag{2.7}$$

$\tilde{\zeta}_i(0) = 0$, $i = 1, \dots, m$, and assume that $(\partial_{\chi_j}\tilde{\zeta}_i(0))_{i,j=1,\dots,n}$ is an invertible (n,n)-matrix

$$(\partial_{\chi_j}\tilde{\zeta}_i(0))_{i,j=1,\dots,n} \in \mathrm{GL}(n), \tag{2.8}$$

and that

$$\partial_\chi\tilde{\zeta}_i(0) = 0 \quad \text{for} \quad i = n+1, \dots, m. \tag{2.9}$$

Let $u^*(\hat{\mu}) = \rho_{\exp(-t\sum_{i=1}^m \zeta_i^*(\hat{\mu})\xi_i)}\Phi_t(u^*(\hat{\mu}))$ be relative equilibria of (2.1) at $\mu_{\mathrm{ext}} = 0$ such that $u^*(0) = u^*$, $\sum_{i=1}^m \zeta_i^*(0)\xi_i = \xi^*$ and $l_i(u^*(\hat{\mu}) - u^*) = 0$, $i = 1, \dots, m$. Then the following holds:

Proposition 2.11 *Under the assumptions of Theorem 2.3 there is a C^k-smooth hyper-surface $(\omega_{\mathrm{ext}}(\mu_{\mathrm{ext}}, \nu), \mu(\mu_{\mathrm{ext}}, \nu))$ of relative periodic orbits $u(\mu_{\mathrm{ext}}, \nu)$ in the $(\omega_{\mathrm{ext}}, \mu)$-parameter-space with $\nu \in \mathbb{R}^d$, $d = p - (\dim(G) - n)$ and $|\nu|$ small, satisfying*

$$\Phi_{2\pi/\omega_{\mathrm{ext}}(\mu_{\mathrm{ext}},\nu),0}(u(\mu_{\mathrm{ext}}, \nu); \mu(\mu_{\mathrm{ext}}, \nu)) = \rho_{\tilde{g}(\chi(\mu_{\mathrm{ext}},\nu))}u(\mu_{\mathrm{ext}}, \nu)$$

and

$$l_i(u(\mu_{\mathrm{ext}}, \nu) - u^*) = 0, \quad i = 1, \dots, m, \quad u(0,0) = u^*, \quad \chi(0,0) = 0,$$

provided that the $(m-n, p)$-matrix

$$(\partial_{(\omega_{\mathrm{ext}},\hat{\mu})}T_{\mathrm{ext}}\zeta_i^*(0))_{i=n+1,\dots,m}$$

has full rank.

Proof. We solve the equation

$$\tilde{g}(\chi)^{-1} g(\omega_{\text{ext}}, \mu) = \text{id}$$

by the implicit function theorem. ∎

In the examples 2.6, 2.8, 2.9 above the hyper-surface $g = \tilde{g}(\chi)$ consists of elements with average drift velocity far away from the drift velocity ξ^* of the relative equilibrium.

2.3 Scaling of drift velocity

In this section we study the scaling of drifts induced by a *harmonic* periodic forcing where the forcing term in (2.1) is of the form

$$f_{\text{ext}}(u, \omega_{\text{ext}} t, \mu) = \tilde{f}(u) \cos(\omega_{\text{ext}} t, \mu). \tag{2.10}$$

Such a forcing term is usually used in experiments [17], [35]. Further let $\mu = \mu_{\text{ext}} \in \mathbb{R}$.

We first state a general proposition, then we apply this result to some examples in spiral wave dynamics explaining scaling laws which were observed in experiments or simulations. In the end we give a mathematical definition of the spiral tip. The motion of the spiral tip is measured in experiments to visualize the drift [5].

We assume that the unforced system (2.1) has a non-critical relative equilibrium u^* and denote again by $\{\xi_1, \xi_2, \ldots, \xi_m\}$ a basis of $\text{alg}(G)$.

Proposition 2.12 *Assume that the periodic forcing term in (2.1) is of the form (2.10). Fix a forcing frequency ω^*_{ext}. Let $u(\mu_{\text{ext}})$, $g(\mu_{\text{ext}})$ be relative periodic orbits for $\mu_{\text{ext}} \approx 0$. Write*

$$g(\mu_{\text{ext}}) = \mathrm{e}^{T_{\text{ext}} \zeta(\mu_{\text{ext}})} \mathrm{e}^{T_{\text{ext}} \xi^*}, \quad \zeta(\mu_{\text{ext}}) = \sum_{i=1}^{m} \zeta_i(\mu_{\text{ext}}) \xi_i.$$

Assume that the geometric multiplicity of the eigenvalue 0 *of the linear map* $[\cdot, \xi^*]$ *on* $\text{alg}(G)$ *equals its algebraic multiplicity. Then*

$$\partial_{\mu_{\text{ext}}} \zeta_i(0) = 0 \quad \text{if} \quad [\xi_i, \xi^*] = 0.$$

This is also true if f_{ext} is not a harmonic periodic forcing, but the mean value $\int_0^{2\pi} f_{\text{ext}}(u, t) \mathrm{d}t$ of f_{ext} is zero.

Now assume that the periodic forcing is resonant so that the linear map $[\cdot,\xi^*]$ *on* $\mathrm{alg}(G)$ *has eigenvalues* $\pm \mathrm{i}\omega_G^*$ *with eigenvectors* $\xi_1 \pm \mathrm{i}\xi_2$ *such that* $\omega_G^*/\omega_{\mathrm{ext}}^* = j \in Z$. *Assume that the algebraic and the geometric multiplicity of the eigenvalue* $\pm\mathrm{i}\omega_G^*$ *of* $[\cdot,\xi^*]$ *are equal. Then*

$$\partial_{\mu_{\mathrm{ext}}}\zeta_i(0) = 0 \quad \textit{for} \quad i=1,2 \quad \textit{if} \quad j>1.$$

If u^* is a rotating wave then $\mathrm{e}^{T_{\mathrm{ext}}\xi^*} = \mathrm{id}$ for some T_{ext}. Therefore the (m,m)-matrix $[\cdot,\xi^*]$ is semisimple and has eigenvalues $\pm\mathrm{i}\omega_G^*$ with $\omega_G^*/\omega_{\mathrm{ext}}^* = j \in Z$ and the above proposition can be applied, see Example 2.13 below.

Proof of Proposition 2.12. We write a prime for $\partial_{\mu_{\mathrm{ext}}}$ in the following calculation. We choose the functionals l_i in (2.5) defining the section transversal to the group orbit again as left center-eigenvectors of L^*. Differentiating (2.5) with respect to μ_{ext} in $\mu_{\mathrm{ext}} = 0$ gives

$$\begin{aligned} \sum_{i=1}^m T_{\mathrm{ext}}^*\zeta_i'(0)\xi_i u^* \;&=\; (\mathrm{e}^{T_{\mathrm{ext}}L^*}-1)u'(0) \\ &\quad +\rho_{\exp(-T_{\mathrm{ext}}\xi^*)}\partial_\mu \Phi_{T_{\mathrm{ext}}^*}(u^*;\mu)|_{\mu=0} \end{aligned} \tag{2.11}$$

where

$$\rho_{\exp(-\xi^* T_{\mathrm{ext}})}\partial_{\mu_{\mathrm{ext}}}\Phi_{T_{\mathrm{ext}}}(u^*;\mu)|_{\mu=0} = \int_0^{2\pi/\omega_{\mathrm{ext}}} \mathrm{e}^{L^*(2\pi/\omega_{\mathrm{ext}}-t)}\tilde f(u^*)\cos(\omega_{\mathrm{ext}}t)\mathrm{d}t.$$

Let P be the spectral projection of L^* to the eigenvalue 0. Since algebraic and geometric multiplicity of the eigenvalue 0 of $[\xi^*,\cdot]$ are equal by assumption and the relative equilibrium u^* is noncritical we conclude that $PL^*=0$ and therefore

$$\int_0^{2\pi/\omega_{\mathrm{ext}}} P\mathrm{e}^{L^*(2\pi/\omega_{\mathrm{ext}}-t)}\tilde f(u^*)\cos(\omega_{\mathrm{ext}}t)\mathrm{d}t = 0.$$

Applying P onto (2.11) we therefore get

$$\sum_{i=1}^m T_{\mathrm{ext}}^*\zeta_i'(0)P\xi_i u^* = \int_0^{2\pi/\omega_{\mathrm{ext}}} P\mathrm{e}^{L^*(2\pi/\omega_{\mathrm{ext}}-t)}\tilde f(u^*)\cos(\omega_{\mathrm{ext}}t)\mathrm{d}t = 0.$$

This proves that $\zeta_i'(0)=0$ if $[\xi_i,\xi^*]=0$, and we see that we get the same result if only the time average of $f_{\mathrm{ext}}(u,t,0)$ is zero.

Now let Q be the spectral projection to the eigenvalue $\mathrm{i}\omega_G^*$, $\omega_G^*/\omega_{\mathrm{ext}}^* = j$. Applying Q onto (2.11) we get, similarly as above,

$$\sum_{i=1}^m T_{\mathrm{ext}}^*\zeta_i'(0)Q\xi_i u^* = \int_0^{2\pi/\omega_{\mathrm{ext}}} Q\mathrm{e}^{L^*(2\pi/\omega_{\mathrm{ext}}-t)}\tilde f(u^*)\cos(\omega_{\mathrm{ext}}t)\mathrm{d}t.$$

As above we conclude that $\zeta_i'(0)=0$ for $i=1,2$ if $j>1$. ∎

Example 2.13 Again let $G = \mathrm{E}(2)$ and let u^* be a non-critical rotating wave of the unforced system (2.1), e.g. a rigidly rotating spiral wave of the reaction-diffusion system (2.2) on the plane $\Omega = \mathbb{R}^2$. Assume that the periodic forcing is resonant $\omega^*_{\text{rot}} = j\omega^*_{\text{ext}}$, $j \in \mathbb{Z}$. Then according to Example 2.6 there is a path $u(\mu_{\text{ext}})$, $a(\mu_{\text{ext}})$, $\omega_{\text{ext}}(\mu_{\text{ext}})$ of modulated travelling waves (drifting spiral waves of the reaction-diffusion system (2.2)) in the parameter-plane $(\omega_{\text{ext}}, \mu_{\text{ext}}) \in \mathbb{R}^2$. Assume that the periodic forcing is harmonic. By Proposition 2.12 the drift velocity $v(\mu_{\text{ext}}) = \frac{a(\mu_{\text{ext}})}{T_{\text{ext}}}$ of the modulated travelling waves satisfies $v'(0) = 0$ if $|j| > 1$.

Drift velocities which only grow with the square μ^2_{ext} of the amplitude of the external periodic forcing are rather small and apparently difficult to find in experiments. That is why in experiments [35] mainly the 1:1-resonance is observed; however in [17] also a $2:1$-resonance could be detected experimentally.

Example 2.14 Let $G = \mathrm{SO}(3)$ and let u^* be a non-critical wave of the unforced system (2.1) rotating around the x_1-axis with speed ω^*_{rot}, for instance, a rigidly rotating spiral wave in the reaction-diffusion system (2.2) on the sphere, see Example 2.8; if the periodic forcing is resonant $\omega^*_{\text{rot}} = j\omega^*_{\text{ext}}$, $j \in \mathbb{Z}$ then according to Example 2.8 there is a path $u(\mu_{\text{ext}})$, $\phi(\mu_{\text{ext}})$, $\omega_{\text{ext}}(\mu_{\text{ext}})$ of modulated rotating waves meandering around some vector in the (x_2, x_3)-plane. By Proposition 2.12 their drift velocity $\omega_{\text{rot}}(\mu_{\text{ext}}) = \phi(\mu_{\text{ext}})/T_{\text{ext}}(\mu_{\text{ext}})$ satisfies $\omega'_{\text{rot}}(0) = 0$ if $j > 1$.

Example 2.15 We again consider a twisted scroll ring, see Example 2.9. In this case the symmetry group is $G = \mathrm{E}(2) \times \mathbb{R}$ and the drift velocity of the scroll ring is given by $\xi^* = (\omega^*_{\text{rot}}, v^*\mathbf{e}_1)$. Denote by $u(\mu_{\text{ext}})$, $g(\mu_{\text{ext}})$ the relative periodic orbits generated by periodic forcing of the twisted scroll with fixed forcing frequency ω_{ext}. We write $g(\mu_{\text{ext}}) = (\phi(\mu_{\text{ext}}), a(\mu_{\text{ext}}))$ where $a(\mu_{\text{ext}}) \in \mathbb{R}^3$, $a(0) = a^*\mathbf{e}_1 = T_{\text{ext}}v^*\mathbf{e}_1$, $\omega_{\text{rot}}(\mu_{\text{ext}}) = \phi(\mu_{\text{ext}})/T_{\text{ext}}$, $\omega_{\text{rot}}(0) = \omega^*_{\text{rot}}$. By Proposition 2.12 we have

$$|\omega_{\text{rot}}(\mu_{\text{ext}}) - \omega^*_{\text{rot}}| = O(\mu^2_{\text{ext}}), \quad |a_1(\mu_{\text{ext}}) - a^*| = O(\mu^2_{\text{ext}}),$$

but in general $|a_i(\mu_{\text{ext}})| = O(\mu_{\text{ext}})$, $i = 2, 3$. This is also observed in numerical simulations, see [15].

Now we define the *tip position* $x_{\text{tip}}(u)$ for $u \in Y$. It is not clear at all how to define the spiral tip exactly. Experimentalists often determine the tip of a spiral wave in two dimensions visually as point with maximal curvature at the end of the spiral [5], but there are also other more or less precise definitions around [14].

From a symmetry point of view the position $x_{\mathrm{tip}}(u) \in \mathbb{R}^2$ of the spiral tip in the case $G = \mathrm{E}(2)$ is a function of the spiral wave solution u into $\mathbb{R}^2$ and has the following property.

Definition 2.16 *The* tip position $x_{\mathrm{tip}}(\cdot)$ *is a C^1-smooth G-equivariant function which maps an open set of Y into a G-manifold M.*

For example in the case $G = \mathrm{E}(2)$ we choose $\pi(\phi, a) = a$, $\pi(G) = \mathbb{R}^2$ and G acts on $\pi(G)$ by the natural affine representation [8]; in the case $G = \mathrm{SO}(3)$ we choose $\pi(G) = S^2$; each $g \in \mathrm{SO}(3)$ can be represented by a vector $\phi \in \mathbf{so}(3) = \mathbb{R}^3$ such that $g = \exp(\phi)$ is a rotation around the unit vector $\phi/|\phi|$ by the rotation angle $|\phi|$; we set $\pi(\exp(\phi)) = \phi/|\phi|$.

In experiments the drift phenomena we talked about are detected by following the spiral tip $x_{\mathrm{tip}}(u)$. For the spiral tip $x_{\mathrm{tip}}(u(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}}))$ the same scaling phenomena hold as for the drifts $g(\omega_{\mathrm{ext}}, \mu_{\mathrm{ext}})$.

2.4 Resonant periodic forcing of relative periodic orbits

Now we consider resonant periodic forcing of relative periodic orbits. We still assume that the isotropy K of the relative periodic orbit is trivial, otherwise we choose $G = N(K)/K$, $Y = \mathrm{Fix}(K)$ as before.

Experiments on periodic forcing of meandering spiral waves have been carried out e.g. by Müller and Zykov [35]. Here invariant 3-tori were found and frequency locking between the period of the relative periodic orbits and the period of the external forcing was observed. Furthermore for certain periods of the external forcing modulated travelling waves were found in experiments. This phenomenon is called "generalized resonance drift" [35].

We will only consider frequency locked relative periodic solutions generated by external periodic forcing. Let again $T_{\mathrm{ext}} = \frac{2\pi}{\omega_{\mathrm{ext}}}$ denote the period of the forcing, let μ_{ext} denote its amplitude and let T^* be the period of the relative periodic orbit for $\mu_{\mathrm{ext}} = 0$. Assume that u^* is a non-critical relative periodic orbit in $\mu = 0$, that is, u^* satisfies $\Phi_{T^*}(u^*) = \rho_{g^*} u^*$, for some $T^* > 0$, $B^* = \rho_{g^*}^{-1} \mathrm{D}\Phi_{T^*}(u^*)$ satisfies Hypothesis (S) and the center-eigenspace only consists of eigenvectors forced by G-symmetry or time-shift symmetry:

$$E_c = \text{alg}(G)u^* \oplus \text{span}(\partial_t \Phi_t(u^*)|_{t=0}).$$

Furthermore suppose that

$$T_{\text{new}} = jT_{\text{ext}} = \ell T^* \quad \text{where} \quad \gcd(j, \ell) = 1.$$

Let P_θ be the spectral projection corresponding to the center spectral set of $\rho_{g^*}^{-1} \text{D}\Phi_1(\Phi_\theta(u^*))$. The conditions $P_\theta(u - \Phi_\theta(u^*)) = 0$ define a section transversal to the relative periodic orbit in $\Phi_\theta(u^*)$.

Proposition 2.17 *Under the above conditions there is a C^k-smooth hypersurface $u(\theta, \mu)$ of $\ell : j$-frequency-locked relative periodic solutions with $\mu \in \mathbb{R}^p$, $\theta \in [0, T^*]$, satisfying*

$$(2.12) \quad \Phi_{\frac{2\pi j}{\omega_{\text{ext}}(\theta,\mu)},0}(u(\theta, \mu)) = \rho_{g(\theta,\mu)} u(\theta, \mu), \quad P_\theta(u(\theta, \mu) - \Phi_\theta(u^*)) = 0,$$

and $u(\theta, 0) = \Phi_\theta(u^*)$, $g(\theta, 0) = (g^*)^\ell$.

This proposition is proved similarly as Theorem 2.3. We refer to section 4 for a proof.

Assume for a moment that G is compact. Due to periodic forcing it may happen that a discrete rotating wave, i.e., a relative periodic orbit u^* for which g^* lies in a discrete Cartan subgroup Z_n, starts drifting. If $\gcd(n, \ell) > 1$, then $(g^*)^\ell$ may lie in a Cartan subgroup $Z_{n_1} \times T^{n/\gcd(n,\ell)}$, $N > 0$ and $\ell : j$-frequency locked relative periodic orbits nearby starts drifting.

An example is the group $G = \text{O}(2)$ where g^* is a reflection. If $\ell = 2$ then modulated rotating waves with relative period $T_{\text{new}} \approx 2T^*$ are generated by the resonant periodic forcing of the discrete rotating wave u^*. Such a phenomenon can not occur in the case of relative equilibria.

Another phenomenon that may occur in the case of periodic forcing is resonance drift as we saw in the preceding sections. Let ξ^* be a drift velocity of g^*. By resonance drift we mean that there are group elements g close to $(g^*)^\ell$ with all average drift velocities ξ far away from the drift velocity ξ^* of g^*. We first give an example. Then we state a general proposition.

Example 2.18 We consider periodic forcing of meandering spiral waves. In this case the symmetry group is $G = \text{E}(2)$, and

$$u^* = \rho_{(-\phi^*,0)} \Phi_{T^*}(u^*)$$

is a modulated rotating wave. Assume that

$$\ell\phi^* = 0 \bmod 2\pi, \quad \ell \neq 0,$$

and that $\hat{\mu} \in \mathbb{R}$ ($p = 2$). If $\partial_{\hat{\mu}}\phi^*(0) \neq 0$ then there is an $\ell : j$-frequency locked modulated travelling wave $u(\theta, \mu_{\rm ext})$ to the parameter $\mu = (\mu_{\rm ext}, \hat{\mu}(\theta, \mu_{\rm ext}),$ $\omega_{\rm ext}(\theta, \mu_{\rm ext})$ such that $u(\theta, 0) = u^*$. Here $\phi^*(\hat{\mu})$ is the rotation angle for the modulated rotating wave $u^*(\hat{\mu}) = \rho_{(-\phi^*(\hat{\mu}),0)}\Phi_{T^*(\hat{\mu})}(u^*(\hat{\mu}))$ for the autonomous system ($\mu_{\rm ext} = 0$) with parameter $\hat{\mu}$.

Let $g = \tilde{g}(\chi)$ as in section 2.1 be a hyper-surface of dimension n in G such that $g(0) = (g^*)^\ell$ and that (2.7), (2.8), (2.9) hold. The hyper-surface $g = \tilde{g}(\chi)$ may for example consist of the group elements with average velocities far away from the drift velocity ξ^* of g^*.

Let $u^*(\hat{\mu}) = \rho^{-1}_{\exp(\sum_{i=1}^m \zeta_i^*(\hat{\mu})T^*(\hat{\mu})\xi_i)g^*}\Phi_{T^*(\hat{\mu})}(u^*(\hat{\mu}))$, $P_0(u^*(\hat{\mu}) - u^*) = 0$, be relative periodic orbits of the unforced system (2.1) where $\mu_{\rm ext} = 0$ such that $u^*(0) = u^*$, $T^*(0) = T^*$, $\zeta_i(0) = 0$, $i = 1, \ldots, m$. Similarly as in Proposition 2.11 we find:

Proposition 2.19 *Under the above assumptions there is a C^k-smooth hyper-surface of $\ell : j$-frequency locked relative periodic orbits near u^* satisfying*

$$\Phi_{\frac{j2\pi}{\omega_{\rm ext}(\theta,\mu_{\rm ext},\nu)},0}(u(\theta, \mu_{\rm ext}, \nu); \mu(\theta, \mu_{\rm ext}, \nu)) = \rho_{\tilde{g}(\chi(\theta,\mu_{\rm ext},\nu))}u(\theta, \mu_{\rm ext}, \nu),$$

and

$$P_\theta(u(\theta, \mu_{\rm ext}, \nu) - \Phi_\theta(u^*)) = 0$$

where $\nu \in \mathbb{R}^d$, $d = p - 1 - (n - \dim(G))$, $|\nu|$ small, provided that the $(n - \dim(G), p - 1)$-matrix

$$(\partial_{\hat{\mu}}\zeta_i^*(0))_{i=n+1,\ldots,\dim(G)}$$

has full rank.

Now we study the scaling behaviour of the drift velocities in the case of harmonic periodic forcing (2.10) which is usually used in experiments [35]. Let $\mu = \mu_{\rm ext} \in \mathbb{R}$.

Proposition 2.20 *Let the periodic forcing be harmonic as in (2.10). Fix a frequency $\omega_{\rm ext}$ of the periodic forcing and write the pull-back elements*

$g(\theta, \mu_{\rm ext})$ of the $\ell : j$-frequency locked periodic orbits, see Proposition 2.17, as

$$g(\theta, \mu_{\rm ext}) = \exp(\sum_{i=1}^{m} jT_{\rm ext}(\theta, \mu_{\rm ext})\zeta_i(\theta, \mu_{\rm ext})\xi_i)(g^*)^\ell.$$

If $\ell > 1$ and if the geometric multiplicity of the eigenvalue 1 of Ad_{g^*} equals its algebraic multiplicity then we have:

$$\partial_{\mu_{\rm ext}}\zeta_i(0) = 0 \quad \textit{for all } i \textit{ with } \mathrm{Ad}_{g^*}\xi_i = \xi_i.$$

Moreover the Arnold tongues where the frequency locking occurs grow as $|\mu_{\rm ext}|^2$.

Note that if $(g^*)^\ell = \mathrm{id}$ as in Example 2.18 the matrix Ad_{g^*} is semisimple so that Proposition 2.20 can be applied.

Again a cautious note: in the case $G = \mathrm{E}(2)$ the meandering spiral wave can not start drifting unboundedly if its spatial symmetry group K contains a nontrivial rotation. In general by periodic forcing the isotropy group of the relative periodic orbit is not changed. So the group element $g(\theta, \mu)$ satisfying $\Phi_{jT_{\rm ext}(\theta,\mu)}(u(\theta, \mu)) = \rho_{g(\theta,\mu)}u(\theta, \mu)$ is in $N(K)$ where K is the isotropy of u^* for properly chosen $u(\theta, \mu)$. Note that we chose $G = N(K)/K$ in the whole section.

Proof of Proposition 2.20. Let $W(t, 0) = \mathrm{D}\Phi_t(u^*)$ denote the solution of the variation equation along $\Phi_t(u^*)$ and let $W(t, s) := W(t, 0)(W(s, 0))^{-1}$, that is, $W(t, s) = \mathrm{D}\Phi_{t-s}(\Phi_s(u_*))$. We have

$$\begin{aligned}
\partial_{\mu_{\rm ext}}\Phi_{T_{\rm new}}(u^*, 0) &= \int_0^{\ell T^*} W(\ell T^*, s)\tilde{f}(\Phi_s(u^*))\cos(\frac{2\pi js}{\ell T^*})\mathrm{d}s \\
&= \int_0^{T^*}(\ldots)\mathrm{d}s + \ldots + \int_{(\ell-1)T^*}^{\ell T^*}(\ldots)\mathrm{d}s \\
&= \mathrm{Re}\left(C\int_0^{T^*} W(T^*, s)\tilde{f}(\Phi_s(u^*))\mathrm{e}^{\frac{2\pi \mathrm{i}js}{\ell T^*}}\mathrm{d}s\right)
\end{aligned}$$

where

$$C = \rho_{g^*}^\ell \sum_{i=0}^{\ell-1}(\rho_{g^*}^{-1}W(T^*, 0))^{\ell-i-1}\mathrm{e}^{2\pi \mathrm{i}ji/\ell}\rho_{g^*}^{-1}.$$

Here we used that

$$\begin{aligned}
W(t + iT^*, s + iT^*) &= \mathrm{D}\Phi_{t-s}(\Phi_{s+iT^*}(u_*)) = \mathrm{D}\Phi_{t-s}(\rho_{g^*}^i\Phi_s(u^*)) \\
&= \rho_{g^*}^i W(t, s)\rho_{g^*}^{-i}
\end{aligned}$$

and that

$$\begin{aligned} W(\ell T^*, iT^*) &= \mathrm{D}\Phi_{(\ell-i)T^*}(\Phi_{iT}(u^*)) = \rho_{g^*}^i \mathrm{D}\Phi_{(\ell-i)T^*}(u^*)\rho_{g^*}^{-i} \\ &= \rho_{g^*}^\ell \rho_{g^*}^{i-\ell} \mathrm{D}\Phi_{(\ell-i)T^*}(u^*)\rho_{g^*}^{-i} \\ &= \rho_{g^*}^\ell (\rho_{g^*}^{-1} W(T^*,0))^{\ell-i}\rho_{g^*}^{-i}, \end{aligned}$$

and that therefore for $s \in [0, T^*)$

$$\begin{aligned} & W(\ell T^*, s+iT^*)\tilde{f}(\Phi_{iT^*+s}(u^*)) \\ &\quad = W(\ell T^*, (i+1)T^*) W(iT^*+T^*, iT^*+s)\rho_{g^*}^i \tilde{f}(\Phi_s(u^*)) \\ &\quad = \rho_{g^*}^\ell (B^*)^{\ell-i-1}\rho_{g^*}^{-1} W(T^*,s)\tilde{f}(\Phi_s(u^*)), \end{aligned}$$

where $B^* := \rho_{g^*}^{-1} W(T^*, 0)$.

Let P_1 be the spectral projection of B^* to the eigenvalue 1. We have

$$P_1 \rho_{g^*}^{-\ell} \partial_{\mu_{\text{ext}}} \Phi_{T_{\text{new}}}(u^*, 0) = \mathrm{Re}\left(c P_1 \rho_{g^*}^{-1} \int_0^{T^*} W(T^*, s)\tilde{f}(\Phi_s(u^*)) \mathrm{e}^{\frac{2\pi \mathrm{i} j s}{\ell T^*}} \mathrm{d}s \right) \tag{2.13}$$

where $c = \sum_{i=0}^{\ell-1} \mathrm{e}^{2\pi \mathrm{i} j i/\ell}$. So $P_1 \partial_{\mu_{\text{ext}}} \Phi_{T_{\text{new}}}(u^*; 0) = 0$ if $\ell > 1$.

Differentiating (2.12) in the solution $(u, g, \omega_{\text{ext}})(\mu)$ with respect to μ_{ext} in $\mu = 0$ yields with $g(\theta, \mu_{\text{ext}})(g^*)^{-\ell} = \exp(\sum_{i=1}^m j T_{\text{ext}}(\theta, \mu_{\text{ext}})\zeta_i(\theta, \mu_{\text{ext}})\xi_i)$

$$\begin{aligned} 0 &= ((B^*)^\ell - 1)\partial_{\mu_{\text{ext}}} u(\theta, \mu_{\text{ext}})|_{\theta, \mu_{\text{ext}}=0} - \ell T^* \sum_{i=1}^m \partial_{\mu_{\text{ext}}} \zeta_i(0)\xi_i u^* \\ &\quad - \frac{2\pi j \partial_{\mu_{\text{ext}}} \omega_{\text{ext}}(0)|}{\omega_{\text{ext}}^2(0)} \partial_t \Phi_t(u^*)|_{t=0} + \rho_{g^*}^{-\ell} \partial_{\mu_{\text{ext}}} \Phi_{T_{\text{new}},0}(u^*, \mu)|_{\mu=0}. \end{aligned}$$

Applying the projection P_1 to the eigenvalue 1 of B^* we see that $\frac{\partial \zeta_i}{\partial_{\mu_{\text{ext}}}}(0) = 0$ for all i with $\mathrm{Ad}_{g^*} \xi_i = \xi_i$ and that $\frac{\partial \omega_{\text{ext}}}{\partial \mu_{\text{ext}}}(0) = 0$ provided that $\ell > 1$.

∎

3 Hopf bifurcation from relative equilibria

In this section we study transitions from relative equlibria to relative periodic orbits in autonomous systems caused by Hopf bifurcation. For experiments on Hopf bifurcation from rotating waves – the meandering transition – in the Belousov-Zhabotinsky reaction see [27], [32], [28]. First we state a general theorem for Hopf bifurcation from relative equilibria. The proof of the Hopf theorem can be found in Subsection 4.6. In Subsection 3.2 we explain the drift phenomena caused by resonance which were observed in experiments. In Subsection 3.3 we discuss equivariant Hopf bifurcation.

In the whole section we assume that the nonlinearity f in (1.3) is autonomous. So we consider the differential equation

$$\frac{\mathrm{d}u}{\mathrm{d}t} = -Au + f(u, \mu). \tag{3.1}$$

In the applications we have in mind (3.1) is an autonomous reaction-diffusion system

$$\frac{\partial u_i}{\partial t} = \delta_i \Delta u_i + f_i(u, \mu), \quad i = 1, \dots, M, \tag{3.2}$$

cf. (1.1).

3.1 The theorem on Hopf bifurcation

Let u^* be a relative equilibrium of (3.1) for $\mu = 0$ satisfying Hypothesis (S). In this subsection we assume that the isotropy K of the relative equilibrium is trivial $K = \{\mathrm{id}\}$ or we exchange G to $N(K)$, Y to $\mathrm{Fix}(K)$. Furthermore suppose that $\pm\mathrm{i}$ are eigenvalues of the linearization $L^* = -A - \xi^* + \mathrm{D}f(u^*)$ in the comoving frame which are not only caused by symmetry, i.e. if Q is the spectral projection of L^* to the i then there is some $w \in QY$ with $w \notin \mathrm{alg}(G)u^*$. Furthermore assume that

$$n\mathrm{i} \in \mathrm{spec}(L^*), n \in \mathbb{Z} \implies QY \subset \mathrm{span}(w, \bar{w}) \oplus \mathrm{alg}(G)u^*.$$

Let $u^*(\mu)$ be the C^k-smooth path of relative equilibria with

$$\Phi_t(u^*(\mu)) = \rho_{\exp(t\xi^*(\mu))}u^*(\mu), \quad l_i(u^*(\mu) - u^*) = 0, \quad i = 1, \dots, m, \quad u(0) = u^*.$$

Note that we can obtain the path of relative equilibria $u^*(\mu)$ near u^* by applying Theorem 2.3 with period $T_{\mathrm{ext}} \notin 2\pi\mathbb{Z}$. As before the functionals l_i, $i = 1, \dots, m$, determine a section transversal to the group orbit of the relative equilibrium u^*.

Lemma 3.1 *There is a C^{k-1}-path $\beta(\mu)$ of eigenvalues of the linearization*

$$L^*(\mu) = -A + \mathrm{D}f(u^*(\mu)) - \xi^*(\mu)$$

such that $\beta(0) = \mathrm{i}$.

This lemma will be proved in section 4.6 below.

We write $\mu = (\mu_1, \mu_2)$ where $\mu_1 \in \mathbb{R}$ and $\mu_2 \in \mathbb{R}^{p-1}$.

Theorem 3.2 *Under the above assumptions there are relative periodic orbits $u(s, \mu_2)$, $s \in \mathbb{R}_0^+$ small, of relative period $T(s, \mu_2)$ near u^* to the parameter $\mu_1(s)$ satisfying*

$$\Phi_{T(s)}(u(s, \mu_2), (\mu_1(s), \mu_2)) = \rho_{g(s,\mu_2)} u(s, \mu_2) \tag{3.3}$$

and $u(0) = u^*$, $\mu(0) = 0$, $g(0) = \mathrm{e}^{2\pi\xi^*}$, $T(0) = 2\pi$ provided that the transversality condition $\frac{\mathrm{d}}{\mathrm{d}\mu_1} \mathrm{Re}\, \beta(0) \neq 0$ is satisfied. For each small s a circle $u(s_1, s_2, \mu_2)$, $s_1 = s \cos t$, $s_2 = s \sin t$, of the relative periodic orbit to the parameter s lies in the section $l_i(u - u^*) = 0$, $i = 1, \dots, m$, and we fix the phase by setting $u(s, \mu_2) = u(s, 0, \mu_2)$, $\partial_s u(0) = Re\, w$. The functions $u(s_1, s_2, \mu_2)$, $\mu_1(s, \mu_2)$, $g(s_1, s_2, \mu_2)$, $T(s, \mu_2)$ are C^{k-1} in $s_1, s_2 \in \mathbb{R}$ and $\mu_2 \in \mathbb{R}^{p-1}$, and $\mu_1(s, \mu_2)$ and $T(s, \mu_2)$ only depend on $s = |s|$ and μ_2.

Theorem 3.2 is proved in section 4.6 below.

The Hopf bifurcation from relative equilibria to relative periodic orbits is called relative Hopf bifurcation because it is a Hopf bifurcation in the space of group orbits. Formally we can compute a semiflow $\Psi_t(\cdot)$ in a comoving frame

$$\Psi_t(u; \mu) = \rho_{g(\Phi_t(u,\mu))} \Phi_t(u + u^*(\mu); \mu) - u^*(\mu) \tag{3.4}$$

where $g(u)$ is such that $l_i(\rho_{g(u)} u - u^*) = 0$, $i = 1, \dots, m$. Under the above assumptions $\Psi_t(\cdot)$ undergoes a usual Hopf bifurcation with two simple Hopf eigenvalues $\pm\mathrm{i}$ and without any resonances. To see this note that the linearization $\mathrm{e}^{\tilde{L}t}$ of $\Psi_t(u)$ in the Hopf point $u = 0$ is given by $\tilde{L} = P_l L^* P_l$ where P_l is the projection onto the space $l_i(u) = 0$, $i = 1, \dots, m$ such that $P_l\, \mathrm{alg}(G) u^* = 0$. But the semiflow $\Psi_t(u)$ is only well-defined on the Banach space Y if the group action is smooth on $\Phi_t(u)$, $t > 0$, which is not the case in applications as we saw in the introduction, cf. subsection 1.5.

Often we need not use the full symmetry G of (3.1) to prove the Hopf theorem. The situation is analogous to the case of periodic forcing of relative equilibria, see section 2.1: If L^* does not have eigenvalues $\mathrm{i}j$, $j \in \mathbb{Z}$, forced by symmetry then $\xi^* = 0$ and we have an ordinary Hopf bifurcation from an equilibrium. If $[\xi^*, \cdot]$ has eigenvalues in $\mathrm{i}\mathbb{Z}$, then the corresponding (generalized) eigenvectors form a Lie subalgebra of $\mathrm{alg}(G)$. We call the group generated by this Lie subalgebra the minimal symmetry group for the Hopf bifurcation.

Example 3.3 Consider again the reaction-diffusion system (3.2) on the domain $\Omega = \mathbb{R}^2$. Then the symmetry group is $G = \mathrm{E}(2)$. Let u^* be a rigidly rotating spiral wave $\Phi_t(u^*) = \rho_{(\omega^*_{\mathrm{rot}} t, 0)} u^*$ of the reaction-diffusion system (3.2). The meandering transition mentioned in the introduction corresponds to a relative Hopf bifurcation from the rotating wave u^*.

3.2 Resonance drift and scaling of drift velocity

In this section we deal with resonant Hopf bifurcation. Again we assume that $K = \{\mathrm{id}\}$ or choose Y as $\mathrm{Fix}(K)$, G as $N(K)/K$. In the next subsection we will deal with equivariant Hopf bifurcation where $K \neq \{\mathrm{id}\}$. Let u^* be a Hopf point with Hopf eigenvalues $\pm\mathrm{i}$, let again $\mu \in \mathbb{R}^p$ and let $u^*(\mu)$ be relative equilibria satisfying $l_i(u^*(\mu) - u^*) = 0$, $i = 1, \ldots, m$, and

$$\Phi_t(u^*(\mu)) = \rho_{\exp(\xi^*(\mu)t)} u^*(\mu), \quad \xi^*(\mu) = \sum_{i=1}^{m} \zeta_i^*(\mu)\xi_i,$$

with $u^*(0) = u^*$, $\xi^*(0) = \xi^*$. Let the assumptions of Theorem 3.2 hold and suppose that $\mathrm{Re}\,\frac{\partial\beta(0)}{\partial\mu_1} \neq 0$. Then we can assume w.l.o.g. that $\mu_1 = 0$ parametrizes the relative equilibria $u^*(\mu)$ which are Hopf points.

We have

$$L^*\xi u^* = [\xi, \xi^*]u^*, \quad \mathrm{e}^{2\pi L^*}\xi u^* = \mathrm{Ad}_{\exp(2\pi\xi^*)}\,\xi u^* = (\mathrm{e}^{2\pi[\cdot,\xi^*]}\xi)u^*, \quad \xi \in \mathrm{alg}(G).$$

If $\exp(\cdot)$ is not locally surjective near $2\pi\xi^*$ then there may be relative periodic orbits bifurcating from the relative equilibrium with all average drift velocities completely different from the drift velocity ξ^* of the relative equilibrium at the Hopf bifurcation. We talk of resonance drift as introduced in subsection 2.1.

For resonance drift to occur it is necessary that the Hopf bifurcation is resonant which means that the linearization L^* of the relative equilibrium in the comoving frame has a symmetry eigenvalue in $\mathrm{i}\mathbb{Z}$. In group-theoretical terms, the linear map $[\cdot, \xi^*]$ has eigenvalues in $\mathrm{i}\mathbb{Z}$. Further ξ^* must not lie in the center of the minimal symmetry group for the Hopf bifurcation. This means that there must be (generalized) eigenvectors of L^* to eigenvalues in $\mathrm{i}\mathbb{Z}$ caused by symmetry which are not eigenvectors to the eigenvalue 0. Otherwise $\exp()$ would be surjective near $2\pi\xi^*$ and the relative periodic orbits $u(s)$ generated by Hopf bifurcation would drift with velocity $\xi(s) \approx \xi^*$.

Let $g = \tilde{g}(\chi)$ be an n-dimensional hyper-surface in G, $\chi \in \mathbb{R}^n$, $|\chi| \leq 1$ such that $\tilde{g}(0) = g^* = e^{\xi^* 2\pi}$. Write $\tilde{g}(\chi) = \exp(\tilde{\zeta}(\chi))g^*$ where $\tilde{\zeta} = \sum_{i=1}^{\dim(G)} \tilde{\zeta}_i(\chi)\xi_i$, $\zeta_i(0) = 0$, $i = 1, \ldots, \dim(G)$, and assume that (2.8) and (2.9) hold. Let $\mu = (\mu_1, \mu_2)$ with $\mu_1 \in \mathbb{R}$, $\mu_2 \in \mathbb{R}^{p-1}$.

Proposition 3.4 *Let the assumptions of Theorem 3.2 and the above assumptions hold and let* $K = \{\mathrm{id}\}$. *If* $\frac{\partial}{\partial\mu_1} \operatorname{Re}\beta(0) \neq 0$ *and if the matrix*

$$\partial_{\mu_2}(\frac{\operatorname{Im}\beta(\mu)}{\zeta_i^*(\mu)})|_{\mu=0}\}_{i=n+1,\ldots,m} \tag{3.5}$$

has full rank then there are relative periodic orbits with average drift inside the hypersurface $g = \tilde{g}(\chi)$, *more precisely: there are* C^{k-1}*-smooth functions* $u(s,\nu)$, $T(s,\nu)$, $\mu(s,\nu)$, $\chi(s,\nu)$ *such that*

$$\Phi_{T(s,\nu)}(u(s,\nu)) = \rho_{\tilde{g}(\chi(s,\nu))}u(s,\nu).$$

Here $\nu \in \mathbb{R}^d$, $d = p - 1 - (\dim(G) - n)$, $\chi(0) = 0$, $u(0) = u^*$.

Proof. By Theorem 3.2 there are relative periodic orbits $u(s,\mu_2)$, $g(s,\mu_2)$, $T(s,\mu_2)$ bifurcating from $u^*(\mu)|_{\mu_1=0}$.

We want to solve the equation $\tilde{g}(\chi)^{-1}g(s,\mu_2) = \mathrm{id}$ by the implicit function theorem. Since $T(0,\mu_2) = \frac{2\pi}{\operatorname{Im}\beta(0,\mu_2)}$ we have

$$\begin{aligned}\partial_{\mu_2} g(0,\mu_2)|_{\mu_2=0} &= \partial_{\mu_2} \exp(\sum_{i=1}^m \frac{2\pi\zeta_i^*(\mu_2)}{\operatorname{Im}\beta(0,\mu_2)}\xi_i) \\ &= 2\pi(\sum_{i=1}^m \partial_{\mu_2}\zeta_i^*(0)\xi_i - \operatorname{Im}\partial_{\mu_2}\beta(0)\xi^*)g^*.\end{aligned}$$

We need that $\partial_{(\chi,\mu_2)}\tilde{g}(\chi)^{-1}g(s,\mu_2)_{(s,\chi,\mu_2)=(0,0,0)}$ has full rank. Therefore the matrix

$$\{\partial_{\mu_2}\zeta_i^*(0) - \zeta_i^*(0)\partial_{\mu_2}\operatorname{Im}\beta(0)\}_{i=n+1,\ldots,\dim(G)}$$

has to be invertible, that is, we need that

$$\{\partial_{\mu_2}(\frac{\operatorname{Im}\beta(\mu)}{\zeta_i^*(\mu)})|_{\mu=0}\}_{i=n+1,\ldots,\dim(G)}$$

has full rank. ∎

Now we study the scaling behaviour of the drift velocities. Let $\mu \in \mathbb{R}$ and write the pull-back elements $g(s)$ of the bifurcating relative periodic orbits $u(s)$ as

$$g(s) = \exp(T(s)\zeta(s))g^*, \quad \zeta(s) = \sum_{i=1}^{\dim(G)} \zeta_i(s)\xi_i. \tag{3.6}$$

Remark 3.5 *Let* $[\xi_i, \xi^*] = 0$. *Then* $\zeta_i'(0) = 0$. *In a* $j : 1$*-resonance*

$$[\xi_1 + \mathrm{i}\xi_2, \xi^*] = \mathrm{i}j(\xi_1 + \mathrm{i}\xi_2)$$

we have $\zeta_i^{(\ell)}(0) = 0$, $i = 1, 2$, $\ell = 1, \ldots, \min(j, k) - 1$.

Proof. Differentiating

$$\rho_{g(s)}^{-1}\Phi_{T(s)}(u(s); \mu(s)) - u(s) = 0$$

w.r.t. s in $s = 0$ gives

$$-2\pi \sum_{i=1}^{m} \zeta_i'(0)\xi_i u^* + (\mathrm{e}^{2\pi L^*} - 1)\,\mathrm{Re}\, w = 0.$$

Applying the spectral projection P_0 of L^* to the eigenvalue 0 gives

$$P_0 \sum_{i=1}^{m} \zeta_i'(0)\xi_i u^* = 0.$$

If $[\xi_i, \xi^*] = 0$ then $P_0\xi_i u^* = \xi_i u^*$, so $\zeta_i'(0) = 0$. We have

$$\Phi_{T(s)}(u(s_1, s_2), \mu(s)) = \rho_{g(s_1,s_2)}u(s_1, s_2) \tag{3.7}$$

where $s_1 = s\cos t$, $s_2 = s\sin t$. By Theorem 3.2 $u(s_1, s_2)$ and $g(s_1, s_2)$ are C^{k-1}-smooth in s_1, s_2. We write $g(s_1, s_2)$ as in (3.6):

$$g(s_1, s_2) = \exp(T(s)\zeta(s_1, s_2))g^*, \quad \zeta(s_1, s_2) = \sum_{i=1}^{\dim(G)} \zeta_i(s_1, s_2)\xi_i.$$

There are C^{k-1}-functions $\hat{g}(t, s) \in G$, $\hat{\zeta}(t, s) \in \mathrm{alg}(G)$ such that $\hat{g}(t, 0) = \mathrm{id}$, $\hat{\zeta}(t, 0) = 0$,

$$\hat{g}(t, s) = \exp(\hat{\zeta}(t, s)), \quad \hat{\zeta}(t, s) = \sum_{i=1}^{m} \hat{\zeta}_i(t, s)\xi_i$$

and

$$u(s_1, s_2) = \rho_{\hat{g}(t,s)\exp(\xi^* tT(s)/2\pi)} \Phi_{tT(s)/2\pi}(u(s), \mu(s)). \tag{3.8}$$

From (3.3), (3.7), (3.8) we conclude that

$$g(s_1, s_2) = \hat{g}(t, s)\exp(\frac{tT(s)}{2\pi}\xi^*)g(s)\exp(-\frac{tT(s)}{2\pi}\xi^*)\hat{g}(t, s)^{-1}.$$

Hence

$$\mathrm{e}^{T(s)\zeta(s_1,s_2)} = \mathrm{e}^{\hat{\zeta}(t,s)}\exp(T(s)\,\mathrm{Ad}_{\exp(\frac{tT(s)}{2\pi}\xi^*)}\,\zeta(s))\mathrm{e}^{-\,\mathrm{Ad}_{g^*}\,\hat{\zeta}(t,s)}.$$

We can choose G minimal such that $\mathrm{Ad}_{g^*} = \mathrm{id}$ on $\mathrm{alg}(G)$. Therefore we conclude that for each i

$$\zeta_i(s_1, s_2)\xi_i = \mathrm{Ad}_{\exp(\hat{\zeta}(t,s))\exp(\frac{tT(s)}{2\pi}\xi^*)}\,\zeta_i(s)\xi_i.$$

Since $\exp(\xi^* t)(\xi_1 + \mathrm{i}\xi_2) = \exp(\mathrm{i}jt)(\xi_1 + \mathrm{i}\xi_2)$ we see that

$$\zeta_1(s_1, s_2)\xi_1 + \mathrm{i}\zeta_2(s_1, s_2)\xi_2 = (1 + O(s))\exp(\mathrm{i}jt)(\zeta_1(s)\xi_1 + \mathrm{i}\zeta_2(s)\xi_2).$$

Therefore since $\zeta_i(s_1, s_2)$, $i = 1, 2$ are C^{k-1}-smooth in s_1, s_2 we conclude that $\zeta_i^{(\ell)}(0) = 0$, $\ell = 0, \ldots, \min(j, k) - 1$, $i = 1, 2$. ∎

Example 3.6 Let again $G = \mathrm{E}(2)$ and let u^* be a rotating wave $\Phi_t(u^*) = \rho_{(\omega^*_{\mathrm{rot}}t,0)}u^*$ of (3.1), e.g. a rigidly rotating spiral wave of (3.2). Assume that the parameter space is two-dimensional, $\mu \in \mathbb{R}^2$, as in Fig. 3, and that parameters are chosen such that the rotating waves $u^*(\mu)$ which are Hopf points lie on the line $\mu_1 = 0$ in parameter space. Note that $\pm\mathrm{i}\omega^*_{\mathrm{rot}}$ are eigenvalues of $[\cdot, \xi^*]$ with eigenvectors $\xi_2 \pm \mathrm{i}\xi_3$, cf. Example 2.1. Because of Proposition 3.4 we now understand Fig. 3: If the rotation frequency ω^*_{rot} is resonant to the Hopf frequency $\omega^*_{\mathrm{Hopf}} = 1$, $\omega^*_{\mathrm{rot}} = j\omega^*_{\mathrm{Hopf}} \in \mathbb{N}$ and the resonance is crossed with nonzero speed $\partial_{\mu_2}(\frac{\mathrm{Im}\,\beta(\mu)}{\omega^*_{\mathrm{rot}}(\mu)})|_{\mu=0} \neq 0$ (which is generically satisfied) then there is a path $\mu(s)$ in parameter space $\mathbb{R}^2$ of modulated travelling waves (drifting spiral waves)

$$\Phi_{T(s)}(u(s); \mu(s)) = \rho_{(0,a(s))}u(s).$$

From Remark 3.5 we see that the drift velocity $v(s) = |a(s)|/T(s)$ generically scales like $|\mu|^{j/2}$, see [4], [8].

3.3 Equivariant relative Hopf bifurcation

In this subsection we study Hopf bifurcation for compact isotropy $K \neq \{\mathrm{id}\}$ of the relative equilibrium u^*. We consider the case when the spatial isotropy K of the relative equilibrium is broken. If the bifurcating solutions are relative periodic solutions and not relative equilibria we talk of equivariant or symmetry-breaking relative Hopf bifurcation.

Assume that the linearization L^* in the relative equilibrium u^* has an eigenvalue i with a generalized eigenvector $w \notin \mathrm{alg}(G)u^*$, i.e. the eigenvalue i of L^* is not (only) caused by symmetry. The generalized eigenspace to the Hopf eigenvalues $\pm$i is K-invariant and may be forced by K-equivariance of L^* to have higher dimension than two even if $\pm$i are not eigenvalues of $[\cdot, \xi^*]$. see [11]. Let P_l denote the projection from Y to the subspace $\{y \mid l(y) = 0\}$. Since K is compact we can choose P_l K-equivariant and $P_l Y = \{y \mid l(y) = 0\}$ K-invariant: for example choose $P = P_s + Q$ where P_s is the projection onto the stable eigenspace of L^* and Q is an orthogonal projection from the finite-dimensional space E_{cu} to $(\mathrm{alg}(G)u^*)^{\perp}$. Since ξ^* commutes with the elements of K the operator $L^* = -A + \mathrm{D}f(u^*) - \xi^*$ is K-equivariant and therefore E_{cu} is invariant and P_s is K-equivariant. If we choose the scalar-product on E_{cu} K-invariant then also Q is K-equivariant. Define $\tilde{L} = P_l L P_l$. Denote the eigenspace of $\tilde{L}$ to the eigenvalues $\pm$i by V. On V the matrices $\mathrm{e}^{\tilde{L}\tau}$, $\tau \in [0, 2\pi]$, define an S^1-action.

We consider the subgroups H of $K \times S^1$ with two-dimensional fixed point spaces. They are called axial subgroups [11]. Let $\pi : K \times S^1 \to K$ be the projection of $K \times S^1$ onto its first component. For each axial subgroup H there is a homomorphism $\Theta : K \to S^1 = \mathbb{R}/\mathbb{Z}$ such that $H = \{(h, \Theta(h) \mid h \in \pi(H)\}$, see [11], [7]. There are two cases, $\Theta(K) = S^1$ or $\Theta(K) = Z_\ell$. Let $\tilde{K}$ denote the kernel of Θ. Then the following lemma holds:

Lemma 3.7 *Let the assumptions of Theorem 3.2 and the above assumptions hold. If $\Theta(K) = S^1$ then there is a symmetry breaking transition from relative equilibria to relative equilibria.*

If $\Theta(K) = Z_\ell$ then a symmetry breaking relative Hopf bifurcation takes place: Let $h^* = \Theta^{-1}(1/\ell) \in K$. There is a path of relative periodic solutions $u(s)$ which emanates from the relative equilibrium u^* by equivariant relative Hopf bifurcation and satisfies

$$\Phi_{T(s)/\ell}(u(s)) = \rho_{g(s)h^*} u(s), \quad T(0) = 2\pi, u(0) = u^*, g(0) = \mathrm{e}^{2\pi\xi^*/\ell}.$$

The isotropy of the bifurcating solutions is $\tilde{K} = \ker(\Theta)$ in both cases.

The proof is a small modification of the proof of Theorem 3.2 and can be found in subsection 4.6. Again the pull-back element $g(s)h^*$ of the relative periodic orbit $u(s)$ has to lie in $N(\tilde{K})$. In the following discussion assume that $G = N(\tilde{K})/\tilde{K}$.

In the case of symmetry breaking Hopf bifurcation the average velocity of the bifurcating relative periodic orbits is often far away from the drift velocity of the relative equilibrium, as we see from the following example.

Example 3.8 (See also [7], [10]) Let again $G = \mathrm{E}(2)$ and let u^* be a rotating wave $\Phi_t(u^*) = \rho_{(\omega_{\mathrm{rot}}^* t, 0)} u^*$ with isotropy $K = Z_\ell$, for example a rigidly rotating spiral wave of (3.2) with ℓ identical arms. Consider a representation of K on the critical eigenspace $V = \mathrm{span}_{\mathbb{C}}(w, \bar{w})$ which is faithful, i.e., $\Theta^{-1}(1/\ell) = 2\pi n/\ell$, $gcd(\ell, n) = 1$. If the rotating wave is a Hopf point then under the usual transversality condition and in the non-resonant case a Hopf bifurcation to modulated rotating waves takes place. The average rotation frequency $\omega_{\mathrm{rot}}(s)$ of the bifurcating modulated rotating waves is given as $\omega_{\mathrm{rot}}(s) = (h^* + \phi(s))/(T(s)/\ell)$. Note that $h^* = 2\pi n/\ell$ and that $g(s) = (\phi(s), a(s))$ satisfies $g(0) = (\omega_{\mathrm{rot}}^* 2\pi/\ell, 0)$. Hence we get

$$\omega_{\mathrm{rot}}(s = 0) = (2\pi n/\ell + \omega_{\mathrm{rot}}^* 2\pi/\ell)/(2\pi/\ell) = n + \omega_{\mathrm{rot}}^*.$$

But in physical space the bifurcating modulated rotating waves in Example 3.8 still seem to drift in a similar direction as the rotating wave u^*. So what is a useful definition of resonance drift in the case of symmetry-breaking Hopf bifurcation? We first continue our example:

Example 3.9 (Example 3.8 continued) We recall the condition for noncompact drift of relative periodic orbits nearby the Hopf point in Example 3.8, see also [7], [10]. Since $g(s) \in N(\tilde{K})$ we can only get noncompact drift if $\tilde{K} \subseteq K$ is trivial. So we consider again as above a faithful representation of K on the critical eigenspace $V = \mathrm{span}_{\mathbb{C}}(w, \bar{w})$, and have $\Theta^{-1}(1/\ell) = 2\pi n/\ell$, $gcd(\ell, n) = 1$. We need $\omega_{\mathrm{rot}}^* = j\ell - n$, $j \in \mathbb{Z}$, since for noncompact drift $\phi(0) = 2\pi\omega_{\mathrm{rot}}^*/\ell + 2\pi n/\ell = 0 \bmod 2\pi$ has to be satisfied. Since $\mathrm{i}\omega_{\mathrm{rot}}^*$ is in the spectrum of $[\cdot, \xi^*]$ with eigenvectors $\xi_1 + \mathrm{i}\xi_2$ we see from Remark 3.5 that the drift velocity $v(s) = |a(s)|/T(s)$ generically grows as $|\mu|^{|j\ell - n|/2}$.

In the case of noncompact drift in the above example we clearly want to speak of resonance drift. Since we do not want to care about the effects of the Z_ℓ-spatio-temporal symmetry of the bifurcating relative periodic orbits in the comoving system (3.4) we only talk of resonance whenever the drift $(g(s)h^*)^\ell$ of the relative periodic orbits after time $T(s)$ is not of the form $\exp(2\pi\xi)$ with $\xi \approx \xi^*$. Note that a necessary condition for resonance drift is that $\exp(\cdot)$ is not locally surjective at $\xi = 2\pi\xi^*$, but since $g(s)$ and h^* need not commute (in contrast to h^* and ξ^*) this condition is not sufficient: in Example 3.6 where the isotropy is trivial the condition for unbounded drift is $\omega^*_{\text{rot}} \in \mathbb{Z}$, in the case of Z_ℓ-isotropy the condition for noncompact drift is more restrictive, see Example 3.9.

4 Proof of the main theorems

This section is devoted to the proof of the theorems on periodic forcing and Hopf bifurcation which we presented in Sections 2 and 3. First, in subsections 4.1 – 4.4 we present a general method how to continue relative periodic orbits that satisfy the spectral hypothesis (S). In subsection 4.5 we prove Theorem 2.3 on periodic forcing. In subsection 4.6 below we use the developed methods to prove the Hopf theorem 3.2 by use of Lyapunov-Schmidt reduction.

4.1 The method of proof

Assume that we are given a relative periodic orbit $u^* = \rho_{g^*}^{-1}\Phi_{2\pi/\omega^*_{\text{ext}},0}(u^*)$ of (2.1) that satisfies the spectral hypothesis (S). We want to continue this relative periodic orbit wrt. the parameters μ and ω_{ext}, i.e., we want to solve the equation $F = 0$ where F is given by

$$F(u, g, \omega_{\text{ext}}, \mu) = \begin{pmatrix} \rho_g^{-1}\Phi_{T_{\text{ext}},0}(u; \omega_{\text{ext}}, \mu) - u \\ l_i(u - u^*), \quad i = 1, \dots, m \end{pmatrix}. \tag{4.1}$$

We consider (4.1) for u in the fixed point space $\text{Fix}(K)$ where K is the isotropy of the relative periodic orbit. W.l.o.g. we assume that $Y = \text{Fix}(K)$ and $G = N(K)$ is the normalizer of K. The functionals l_i, $i = 1, \dots, m$, define a section transversal to the group orbit $Gu^* = \{\rho_g u^* \mid g \in G\}$ at u^*. We can not solve (4.1) by the ordinary implicit function theorem because in general $F(u, g, \omega_{\text{ext}}, \mu)$ is only continuous in g. This comes from the fact that the G-action is only strongly continuous and the Lie algebra elements $\xi \in \text{alg}(G)$ act in general as unbounded operators on Y. Furthermore, the time-evolution does not smoothen the group action, that is, $\rho_g\Phi_{T_{\text{ext}},0}(u)$ is not differentiable in g in general. This is due to the fact that the operators $\xi \in \text{alg}(G)$ are not assumed to be bounded w.r.t. A (in the case of the reaction-diffusion system (1.1) the operator $\frac{\partial}{\partial\phi}$ is not bounded w.r.t. Δ, see Proposition 1.2). Therefore the operator $\frac{\partial F}{\partial u}(u, g, \omega_{\text{ext}}, \mu)$ is in general not continuous in g with respect to the norm $\|\cdot\|_{\mathcal{L}(Y)}$. We overcome these difficulties as follows:

We will solve the fixed point equation

$$(4.2) \qquad y = \Pi(y, q, g, \omega_{\text{ext}}, \mu) = (1 - \hat{P})\rho_g^{-1}\Phi_{T_{\text{ext}},0}(y + q; \omega_{\text{ext}}, \mu),$$

$y \in (1 - \hat{P})Y$, $q \in \hat{P}Y$, by Banach's contraction mapping theorem. Here $\hat{P}$ is a projector which is near the projection P onto the center-unstable eigenspace E_{cu} of $B^* = \rho_{g^*}^{-1}\mathrm{D}\Phi_{T_{\text{ext}},0}(u^*)$ in the $\mathcal{L}(Y)$-norm. Furthermore we will show that the solution $y(q, g, \omega_{\text{ext}}, \mu)$ of this fixed point equation depends smoothly on the parameters $(q, g, \omega_{\text{ext}}, \mu)$ and that the G-action on the solutions is smooth. Then we plan to solve the reduced equation $F_{\text{red}} = 0$

$$(4.3) \quad F_{\text{red}}(q, g, \omega_{\text{ext}}, \mu) = \begin{pmatrix} \hat{P}\rho_g^{-1}\Phi_{T_{\text{ext}},0}(y(q, g, \omega_{\text{ext}}, \mu) + q; \omega_{\text{ext}}, \mu) - q \\ l_i(y(q, g, \omega_{\text{ext}}, \mu) + q - u^*) = 0, \quad i = 1, \ldots, m \end{pmatrix}$$

by the implicit function theorem. In this way we can solve (4.1). Note that we need $\rho_g u^*$ to be differentiable in g in order to define the functionals l_i, $i = 1, \ldots, m$.

4.2 The scale of Banach spaces $\{Y_j\}_{j=0,\ldots,k}$

For $j > 1$, define inductively

$$(4.4) \qquad Y_j := \{u \in Y_{j-1};\ \xi u \in Y_{j-1} \text{ for any } \xi \in \operatorname{alg}(G)\}, \quad Y_0 = Y,$$

equipped with the graph norm $|\cdot|_{Y_j}$ given by

$$|u|_{Y_j} = |u|_{Y_{j-1}} + \sup_{\xi \in \operatorname{alg}(G), |\xi|=1} |\xi u|_{Y_{j-1}}.$$

Let $Y^\star$ be the dual space to Y and define

$$Z_0^\star := \{y^\star \in Y^\star;\ \rho_g^\star y^\star \text{ is } C^0 \text{ in } g\},$$

where $\rho_g^\star$ denotes the adjoint operator of ρ_g in $Y^\star$. For $j > 1$, we define the spaces $Z_j^\star$ with norm $|\cdot|_{Z_j^\star}$ for the adjoint group action as in (4.4) with Y_0 replaced by $Z_0^\star$.

In the following we will often use that $P\rho_g$ and $\rho_g P$ are continuous in g with respect to the norm $\|\cdot\|_{\mathcal{L}(Y)}$. For the second operator this is clear since ρ_g is strongly continuous in g and PY is finite-dimensional. The operator $P\rho_g$ is continuous in g with respect to the norm $\|\cdot\|_{\mathcal{L}(Y)}$ iff $\rho_g^\star P^\star$ is continuous with respect to the norm $\|\cdot\|_{\mathcal{L}(Y^\star)}$ where $P^\star$ is the spectral projection in $Y^\star$ onto the left center-unstable eigenspace of L^*.

Lemma 4.1 *$P^\star$ maps $Y^\star$ into $Z_0^\star$.*

If the group G acts strongly continuously on the dual space, for example in the case $G = \mathrm{E}(2)$ acting on $Y = L^2(\mathbb{R}^2, \mathbb{R}^M)$, then Lemma 4.1 is automatically satisfied. Therefore we will skip the proof which is elementary, but technical and can be found in [24, Lemmata 5.1,5.2].

Remark 4.2 *If we replace the assumption of a C^0-action of G by the assumption that $\rho_g u^*$ is continuous in g and the group action is weakly continuous then Theorem 2.3 still holds.*

This is due to the fact that $PY \subset Y_0$ is still satisfied, see [24, Lemmata 5.1,5.2] and we can therefore restrict the problem onto Y_0.

Since $\xi\Phi_{t,t_0}(u) = \mathrm{D}\Phi_{t,t_0}(u)\xi u$ we see that Φ_{t,t_0} maps Y_1 into Y_1. Inductively we see that the time-evolution Φ_{t,t_0} maps each Y_j, $j \leq k$, into itself. Further Φ_{t,t_0} is C^{k-j}-smooth from Y_j into Y_j.

Now we need the following lemma:

Lemma 4.3 *Y_1 is dense in Y_0 and $Z_1^\star$ is dense in $Z_0^\star$. Moreover, G acts as C^0-group on Y_j, $Z_j^\star$.*

The proof can be found in [24, Lemma 4.1]. If $\dim(G) = 1$ this is usual semigroup theory. From this lemma we can deduce

Lemma 4.4 *There is a projector $\hat{P}$ near P such that $\rho_g \hat{P}$ and $\hat{P}\rho_g$ are C^k in g.*

This was shown in [24, Lemma 5.3]. The idea is the following: let e_i, $i = 1, \ldots, \dim(PY)$ be a basis of PY, and $e_i^\star$, $i = 1, \ldots, \dim(PY)$, be a basis for $P^\star Y$. Then by the foregoing lemma we can find $\hat{e}_i \in Y_k$, $\hat{e}_i^\star \in Z_k^\star$ which are near e_i rsp. $e_i^\star$ in the Y-norm rsp. $Y^\star$-norm. From these vectors $\hat{e}_i$, $\hat{e}_i^\star$ we "build" the projection $\hat{P}$.

4.3 Regularity of the relative periodic orbit

Now we need the following main lemma which will inductively yield C^k-regularity of Gu^* and $\rho_g P$, $P\rho_g$:

Lemma 4.5 *If Hypothesis (S) is satisfied then $u^* \in Y_1$.*

Proof. For a proof involving exponential dichotomies see [24]. Here we will give a more elementary proof.

In a first step we define a formal expression for ξu^*, $\xi \in \operatorname{alg} G$, and in a second step we will show that ξu^* exists and indeed equals this expression.

Let $\hat{P}$ be a projector near P such that $\rho_g \hat{P}$ and $\hat{P}\rho_g$ are C^1 in g in the operator norm on Y. Since $u^* = \rho_{g^*}^{-1}\Phi(u^*)$ and $\xi\rho_g^{-1} = \rho_g^{-1}\operatorname{Ad}_g \xi$ we have

$$\xi u^* = \rho_{g^*}^{-1}(\operatorname{Ad}_{g^*}\xi)\Phi(u^*) = \rho_{g^*}^{-1}\mathrm{D}\Phi(u^*)(\operatorname{Ad}_{g^*}\xi)u^* = B^*(\operatorname{Ad}_{g^*}\xi)u^*$$

where $B^* = \rho_{g^*}^{-1}\mathrm{D}\Phi(u^*)$ and so we formally get

$$z(\xi) = B_s z(\operatorname{Ad}_{g^*}\xi) + \eta(\xi). \tag{4.5}$$

Here

$$z(\xi) := (1-\hat{P})\xi u^*, \quad B_s := (1-\hat{P})B^*, \quad \eta(\xi) = B_s\hat{P}(\operatorname{Ad}_{g^*}\xi)u^*.$$

Note that $z(\xi)$ and $\eta(\xi)$ are linear in ξ. Since $\hat{P}$ is near P and since the spectral radius of $(1-P)B^*$ is smaller than one also the spectral radius of B_s is smaller than one. Let $\{\xi_i, i = 1,\ldots,m\}$ be a basis of $\operatorname{alg}(G)$. By our overall hypothesis the operator $\operatorname{Ad}_{g^*} : \operatorname{alg}(G) \to \operatorname{alg}(G)$ has spectrum on the unit circle. Let $(\operatorname{Ad}_{g^*})_{ij}$ be the matrix associated to the operator Ad_{g^*} with respect to the basis $\{\xi_i, i = 1,\ldots,m\}$ of $\operatorname{alg}(G)$. We can define Ad_{g^*} as operator in $Y^m = Y \times \ldots \times Y$ by setting

$$\operatorname{Ad}_{g^*}(z_1,\ldots,z_m) := (s_1,\ldots,s_m), \quad s_i = \sum_{j=1}^m (\operatorname{Ad}_{g^*})_{ij} z_j.$$

Also the operator B_s can be extended to an operator on Y^m by defining

$$B_s(z_1,\ldots,z_m) := (B_s z_1,\ldots,B_s z_m), \quad z_i \in Y, \quad i = 1,\ldots,m.$$

Hence the operator $B_s\operatorname{Ad}_{g^*} = \operatorname{Ad}_{g^*}B_s$ on Y^m has also spectral radius smaller than one. Changing $\Phi_{T_{\mathrm{ext}},0}$ to $\Phi_{\ell T_{\mathrm{ext}},0}$ and accordingly B^* to $(B^*)^\ell$ and g^* to

$(g^*)^\ell$ with ℓ large enough we can achieve that $\|B_s\||\mathrm{Ad}_{g^*}| < 1$. W.l.o.g. we assume that $\ell = 1$. We rewrite (4.5) as

$$(4.6) \qquad (1 - B_s \mathrm{Ad}_{g^*}) z = \eta, \quad z = (z_1, \ldots, z_m), \quad \eta = (\eta_1, \ldots, \eta_m)$$

where $\eta_i = \eta(\xi_i)$, $i = 1, \ldots, m$, are well-defined since $\hat{P}\rho_g$ is C^1 in g in the operator norm. The system of equations (4.6) can be solved uniquely for $z_i = z(\xi_i)$, $i = 1, \ldots, m$. So we have proved that $\xi_i u^* = z_i + \hat{P}\xi_i u^*$ formally exists for all ξ_i, $i = 1, \ldots, m$, and hence by linear combination we get for each $\xi \in \mathrm{alg}(G)$ a formal expression $z(\xi) + \hat{P}\xi u^*$ which we know equals ξu^* if $u^* \in Y_1$.

To show that the formal expression $z(\xi)$ is indeed $(1 - \hat{P})\xi u^*$ we argue as follows. Let $z(\xi, t) = \frac{1}{t}(1 - \hat{P})(\rho_{\exp(\xi t)} u^* - u^*)$. We have

$$(4.7) \qquad \begin{aligned} z(\xi, t) &= \tfrac{1}{t}(1 - \hat{P})(\rho_{\exp(\xi t)} \rho_{g^*}^{-1} \Phi(u^*) - \rho_{g^*}^{-1} \Phi(u^*)) \\ &= B_s(t) z(\mathrm{Ad}_{g^*} \xi, t) + \eta(\xi, t) \end{aligned}$$

where

$$B_s(t) := (1 - \hat{P}) \rho_{g^*}^{-1} D\Phi(u^* + \Theta(t)(\rho_{\exp(\mathrm{Ad}_{g^*} \xi t)} u^* - u^*))$$

with $0 \le \Theta(t) \le 1$ and

$$\eta(\xi, t) = \frac{1}{t} B_s(t) \hat{P}(\rho_{\exp(\mathrm{Ad}_{g^*} \xi t)} u^* - u^*).$$

Here we applied the mean value theorem. Let $\delta_z(\xi, t) = z(\xi, t) - z(\xi)$. Then

$$(4.8) \qquad \delta_z(\xi, t) = B_s(t) \delta_z(\mathrm{Ad}_{g^*} \xi, t) + \delta_\eta(\xi, t)$$

where

$$\delta_\eta(\xi, t) = (B_s(t) - B_s) z(\xi) + \eta(\xi, t) - \eta(\xi)$$

converges to zero as $t \to 0$. Let $\epsilon_z(t) = \sup_{|\xi| \le 1, |\tau| \le t} \delta_z(\xi, \tau)$, $\epsilon_\eta(t) = \sup_{|\xi| \le 1, |\tau| \le t} \delta_\eta(\xi, \tau)$. Here $|\xi| = (\sum_{i=1}^m \zeta_i^2)^{1/2}$ for $\xi = \sum_{i=1}^m \zeta_i \xi_i$ is a norm on $\mathrm{alg}\, G$. We define $B_s(t)$ like B_s as operator from Y^m into Y^m. Then $\|B_s(t)\||\mathrm{Ad}_{g^*}| = c < 1$ for t small enough since $B_s(t)$ is continuous in t in the $\mathcal{L}(Y)$-norm and $\|B_s\||\,\mathrm{Ad}_{g^*}\,| < 1$.

From (4.8) we get

$$(4.9) \qquad \epsilon_z(t) \le c \epsilon_z(|\mathrm{Ad}_{g^*}| t) + \epsilon_\eta(t)$$

with $\epsilon_\eta(t) \to 0$ as $t \to 0$. Here we used that

$$\begin{aligned} z(\mathrm{Ad}_{g^*}\xi, t) &= z(\frac{1}{|\mathrm{Ad}_{g^*}|}\mathrm{Ad}_{g^*}\xi, |\mathrm{Ad}_{g^*}|t)|\mathrm{Ad}_{g^*}|, \\ z(\mathrm{Ad}_{g^*}\xi) &= z(\frac{1}{|\mathrm{Ad}_{g^*}|}\mathrm{Ad}_{g^*}\xi)|\mathrm{Ad}_{g^*}| \end{aligned}$$

and that therefore

$$\delta_z(\mathrm{Ad}_{g^*}\xi, t) = \delta_z(\frac{1}{|\mathrm{Ad}_{g^*}|}\mathrm{Ad}_{g^*}\xi, |\mathrm{Ad}_{g^*}|t)|\mathrm{Ad}_{g^*}|$$

and consequently

$$\sup_{|\xi|\leq 1} \delta_z(\mathrm{Ad}_{g^*}\xi, t) \leq |\mathrm{Ad}_{g^*}| \sup_{|\xi|\leq 1} \delta_z(\xi, |\mathrm{Ad}_{g^*}|t).$$

From (4.9) we conclude that

$$\epsilon_z(t) \leq c^\ell \epsilon_z(|\,\mathrm{Ad}_{g^*}|^\ell t) + \sum_{i=0}^{\ell-1} c^i \epsilon_\eta(|\,\mathrm{Ad}_{g^*}|^i t),$$

and hence that

$$\epsilon_z(t/|\,\mathrm{Ad}_{g^*}|^\ell) \leq \frac{1-c^\ell}{1-c}\epsilon_\eta(t) + c^\ell \epsilon_z(t).$$

Choosing t small enough and ℓ large enough we see that $\epsilon_z(t) \to 0$ as $t \to 0$.

■

4.4 Contractions on a scale of Banach spaces

We first show (Lemma 4.6) that Π is a contraction in $(1-\hat{P})Y$. Afterwards, in Lemma 4.7, we show that we can apply the contraction theorem on the Banach scale $\{(1-\hat{P})Y_j\}_{j=0,\ldots,k-1}$. Finally Theorem 4.8 below guarantees that the solution we obtained depends smoothly on parameters.

Lemma 4.6 *Let u^* be a relative periodic orbit of (1.3) to the parameters $(\omega^*_{\mathrm{ext}}, \mu^*)$ fulfilling the spectral condition (S). Let $\hat{P}$ be a projection which is $\mathcal{L}(Y)$-near P. Let $(g, \omega_{\mathrm{ext}}, \mu)$ be near $(g^*, \omega^*_{\mathrm{ext}}, \mu^*)$ and let $(y+q)$ be near (y^*+q^*) in the Y-norm with $y^*, y \in (1-\hat{P})Y$, $q, q^* \in \hat{P}Y$, $q^*+y^* = u^*$. Then Π satisfies*

$$\|\frac{\partial \Pi^\ell}{\partial y}(y, q, g, \omega_{\mathrm{ext}}, \mu)\| \leq c < 1,$$

where $\ell \in \mathbb{N}$ is sufficiently large.

Proof of Lemma 4.6. We have $\|(B^*(1-P))^\ell\| \le MC^\ell$, $C<1$. Let $\ell \in \mathbb{N}$ be so large that

$$\|(1-P)\rho_g(B^*)^\ell(1-P)\| \le \|(1-P)\|MC^\ell M_G < 1.$$

Here we used that for g in a neighborhood of id there is a uniform bound M_G of $\|\rho_g\|$. Then $(1-P)\rho_g(B^*)^\ell(1-P)$ is a uniform contraction for g near id. We have

$$\mathrm{D}_y\Pi^\ell(y) = \prod_{i=0}^{\ell-1}\mathrm{D}\Pi(\Pi^i(y)) = \prod_{i=0}^{\ell-1}(1-\hat{P})\rho_g^{-1}\mathrm{D}_y\Phi_{T_{\mathrm{ext}},0}(q+\Pi^i(y)).$$

Since y is near y^*, q is near q^* and g is near g^* we know that $\Pi^i(y)\approx y^*$ and that

$$\rho_g^{-1}\mathrm{D}_u\Phi_{T_{\mathrm{ext}},0}(q+\Pi^i(y)) \approx \rho_g^{-1}\rho_{g^*}B^*$$

in the operator norm. Therefore we conclude that

$$\mathrm{D}_y\Pi^\ell(y) \approx \prod_{i=0}^{\ell-1}(1-P)\rho_{g^{-1}g^*}B^*(1-P)$$

in the norm on $\mathcal{L}(Y)$. Further we compute

$$\begin{aligned}(\rho_{g^{-1}g^*}B^*)^2 &= \rho_{g^{-1}}\mathrm{D}_y\Phi_{T_{\mathrm{ext}},0}(u^*)\rho_{g^{-1}g^*}B^* \approx \rho_{g^{-1}}\rho_{g^{-1}g^*}\mathrm{D}_y\Phi_{T_{\mathrm{ext}},0}(u^*)B^* \\ &= \rho_{g^{-2}(g^*)^2}(B^*)^2.\end{aligned}$$

Similarly we get

$$(\rho_{g^{-1}g^*}B^*)^\ell \approx \rho_{g^{-\ell}(g^*)^\ell}(B^*)^\ell.$$

Since $\rho_g\hat{P}$ and $\hat{P}\rho_g$ are continuous in g in the operator norm and since $\hat{P}$ is near P in the $\|\cdot\|_{\mathcal{L}(Y)}$-norm we conclude that $\mathrm{D}_y\Pi^\ell(y)$ is near $(1-P)\rho_{g^{-\ell}(g^*)^\ell}(B^*)^\ell(1-P)$ for g near g^*, y near y^*, q near q^* in the operator norm. Hence $\frac{\partial\Pi^\ell}{\partial y}(y,q,g,\omega_{\mathrm{ext}},\mu)$ is a contraction if we choose $(y+q,g,\omega_{\mathrm{ext}},\mu)$ near $(y^*+q^*,g^*,\omega^*_{\mathrm{ext}},\mu^*)$ (here we measure $y-y^*$, $q-q^*$ in the Y-norm). ∎

Now we show that Π is a contraction on the scale of Banach spaces $\{(1-\hat{P})Y_j\}_{j=0,\dots,k-1}$.

Lemma 4.7 *Let u^* be a relative periodic orbit of (1.3) to the parameters $(\omega^*_{\mathrm{ext}},\mu^*)$ fulfilling Hypothesis (S). If f is C^k-smooth, $k\in\mathbb{N}$, then we have:*

(i) $u^* \in Y_k$.

(ii) $B^*Y_j \subseteq Y_j$, $(B^*)^\star Z_j^\star \subseteq Z_j^\star$ *and* $\mathrm{spec}(B_j^*) \subset \mathrm{spec}(B^*)$, $j = 1, \ldots, k-1$, *where* B_j^* *is the operator* B^* *considered as map from* Y_j *into itself. Further,* $P \in \mathcal{L}(Y, Y_{k-1})$, $P^\star \in \mathcal{L}(Y^\star, Z_{k-1}^\star)$.

(iii) u^* *satisfies Hypothesis (S) on each* Y_j, $0 \leq j \leq k-1$.

(iv) Let $\hat{P}$ *be* $\mathcal{L}(Y, Y_{k-1})$*-near* P. *If* $\ell = \ell(k) \in \mathbb{N}$ *is large enough the function* $y \to \Pi^\ell(y, q, g, \omega_{\mathrm{ext}}, \mu)$ *from (4.2) is a uniform contraction on each* Y_j, $0 \leq j \leq k-1$, *for* $y + q$ Y_j*-near* u^* *and* $(g, \omega_{\mathrm{ext}}, \mu)$ *near* $(g^*, \omega_{\mathrm{ext}}^*, \mu^*)$.

(v) Let $\hat{P}$ *be as in (iv) and assume that* $\rho_g \hat{P}$ *and* $\hat{P}\rho_g$ *are* C^k*-smooth in the* $\mathcal{L}(Y)$*-norm. Then there is a locally unique solution* $y(q, g, \omega_{\mathrm{ext}}, \mu) \in (1-\hat{P})Y$ *of (4.2) which is continuous in* $(q, g, \omega_{\mathrm{ext}}, \mu)$ *with respect to the norm* $\|\cdot\|_{Y_k}$.

Part (i) of this lemma can also be found in [24].

Proof of Lemma 4.7. Suppose that $u^* \in Y_j$ for some j with $j \geq 1$, $j < k$. Since Φ_{t,t_0} is a time-evolution on each Y_j and G acts as C^0-group on each Y_j w.r.t. the Y_j-norm by Lemma 4.3 we know that $B^* \in \mathcal{L}(Y_i)$, $i \leq j$. We have

$$\xi(B^* - \lambda) = (B^* - \lambda)\,\mathrm{Ad}_{g^*}\,\xi + V(\xi),$$

with

$$V(\xi) := \partial_u^2 \Phi_{T_{\mathrm{ext}},0}(u^*)(\mathrm{Ad}_{g^*}\,\xi)u^* \in \mathcal{L}(Y_{j-1}).$$

Let $\lambda \in \mathbb{C} \setminus \mathrm{spec}(B^*)$ lie in the resolvent set of B^*. Then we get

$$\mathrm{Ad}_{g^*}\,\xi(B^* - \lambda)^{-1} = (B^* - \lambda)^{-1}\xi - (B^* - \lambda)^{-1}V(\xi)(B^* - \lambda)^{-1}. \tag{4.10}$$

Let B_j be the operator B^* considered as element of $\mathcal{L}(Y_j)$. From (4.10) we deduce that $\mathrm{spec}(B_j^*) \subset \mathrm{spec}(B_{j-1}^*) \subset \ldots \subset \mathrm{spec}(B_0^*)$. Let σ be the spectral set of the center-unstable eigenvalues of B^*. Then

$$P = \frac{1}{2\pi \mathrm{i}} \oint_{\text{around } \sigma} (\lambda - B^*)^{-1} \mathrm{d}\lambda. \tag{4.11}$$

From (4.11) we see that P maps Y_j into itself if $u^* \in Y_j$. Since Y_j is dense in Y by iterative application of Lemma 4.3 we can find $w_i \in Y_j$, $i = 1, \ldots \dim(PY)$, which span PY. Hence $PY \subseteq Y_j$. Since $\xi u^* \in PY$, $\xi \in \operatorname{alg}(G)$, we infer $u^* \in Y_{j+1}$.

According to Lemma 4.5 we have $u^* \in Y_1$ if $k \geq 1$. Hence by induction we obtain

$$u^* \in Y_k, \quad PY \subseteq Y_{k-1}.$$

By computing the adjoints on both sides of equation (4.10) we see that $B^\star Z_j^\star \subset Z_j^\star$, $0 \leq j \leq k-1$. Analogously as above we obtain $P^\star Y^\star \subset Z_{k-1}^\star$. Using (i) and (ii) we conclude that u^* satisfies condition (S) on each Y_j, $j \leq k-1$.

To prove (iv) we apply Lemma 4.6 on each Y_j, $j \leq k-1$. Applying the contraction principle on each Y_j, $j \leq k-1$, we obtain solutions $y_j(q, g, \omega_{\text{ext}}, \mu)$ of $y = \Pi^\ell(y)$ which are continuous in the parameters and locally unique in Y_j and therefore solutions of (4.2). Since $Y_j \subseteq Y$ for all j the solutions all equal the solution $y(q, g, \omega_{\text{ext}}, \mu)$.

In the same way as in Lemma 4.5 we can show that $y = y(q, g, \omega_{\text{ext}}, \mu) \in Y_k$: Assume for simplicity that $\ell = 1$, otherwise the formulas get more complicated. From $y = \Pi(y)$ we formally get the identity

$$z(\xi) = B_s z(\operatorname{Ad}_{g^*} \xi) + \eta(\xi)$$

on Y_{k-1} where $z(\xi) = (1 - \hat{P})\xi y(q, g, \omega_{\text{ext}}, \mu)$, $B_s = \mathrm{D}\Pi(q + y)(1 - \hat{P})$ and

$$\eta(\xi) = -(1 - \hat{P})\xi \hat{P} \rho_g^{-1} \Phi_{T_{\text{ext}},0}(y + q) + \mathrm{D}\Pi(y + q)(\hat{P} \operatorname{Ad}_{g^*} \xi y + \operatorname{Ad}_{g^*} \xi q).$$

The operator $\eta(\xi)$ is for $y \in Y$ because $\hat{P}\rho_g$ and $\hat{P}\rho_g$ are C^k-smooth in the $\mathcal{L}(Y)$-norm. Since B_s is a contraction this equation can be solved uniquely for $z(\xi_i)$, $i = 1, \ldots, m$. In the same way as in the proof of Lemma 4.5 we can now show that the formal derivative $z(\xi) + \hat{P}\xi y(q, g, \omega_{\text{ext}}, \mu)$ is indeed the derivative $\xi y(q, g, \omega_{\text{ext}}, \mu)$. We infer that $y(q, g, \omega_{\text{ext}}, \mu)$ is continuous in its parameters in the norm of Y_k. ∎

In order to show that the solutions really depend C^k-smoothly on their parameters we will use a contraction mapping theorem on a scale of Banach spaces. This idea has frequently been used in the literature, for example it is used to prove the smoothness of center manifolds (Vanderbauwhede & Van Gils [31], Vanderbauwhede & Iooss [30]). Renardy [20] proved a generalized implicit function theorem on a scale of Banach spaces $\{Y_j\}_{0 \leq j \leq k}$; he required that the derivative of the nonlinear equation to be solved evaluated at the starting solution depends continuously on the parameter with respect to the norm $\|\cdot\|_{\mathcal{L}(Y_j)}$. As in [31] we will assume that the derivative is a contraction. Hard implicit function theorems can be found in Nirenberg [18]. We will employ the following theorem which is stated in general form in [31] for $k = 1$.

Theorem 4.8 *Let $\mathcal{Y} = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \ldots \supset \mathcal{Y}_k$, $k \geq 1$, be a scale of Banach spaces with norms $\|\cdot\|_{\mathcal{Y}_j}$, $j \leq k$, and let $\mathcal{Y}_j$ be continuously embedded in $\mathcal{Y}_{j-1}$. Let $(u, \nu) \to \Pi(u, \nu)$ be a nonlinear map from some open set $U \subset \mathcal{Y} \times \mathbb{R}^p$ into $\mathcal{Y}$. Assume the following:*

(i) Π *maps* $U_j := (\mathcal{Y}_j \times \mathbb{R}^p) \cap U$ *into* $\mathcal{Y}_j$ *and* Π *is* $C^{\ell-j}$*-smooth from* U_ℓ *to* $\mathcal{Y}_j$, $j, \ell \in \mathbb{N}_0$, $k \geq \ell \geq j \geq 0$.

(ii) $(\nu, u, w_1, \ldots, w_j) \to \frac{\partial^{j+\ell}}{\partial u^j \partial \nu^\ell} \Pi(u, \nu)(w_1, \ldots, w_j)$ *is continuous as map from* $U_i \times (\mathcal{Y}_i)^j$ *into* $\mathcal{L}^\ell(\mathbb{R}^p, \mathcal{Y}_{i-\ell})$, *for* $i, j, \ell \in \mathbb{N}_0$, $\ell \leq i \leq k$, $j + \ell \leq k$, *where* $\mathcal{L}^0(\mathbb{R}^p, \mathcal{Y}_i) := \mathcal{Y}_i$.

(iii) $\Pi(\cdot, \nu)$ *is a uniform contraction as map from* U_j *into* $\mathcal{Y}_j$, $0 \leq j \leq k-1$, *with contraction constant* $c < 1$.

Then

a) there is a unique solution $u(\nu) \in \mathcal{Y}_{k-1}$ *to* $\Pi(u, \nu) = u$ *and* $u(\nu)$ *is a* C^{k-1}*-function of* ν *with respect to the norm* $\|\cdot\|_{\mathcal{Y}}$.

b) If we require in addition

(iv) $u(\nu)$ *is continuous in the norm* $\|\cdot\|_{\mathcal{Y}_k}$

then $u(\nu)$ *is a* C^k*-function of* ν *with respect to the norm* $\|\cdot\|_{\mathcal{Y}}$.

Proof. We can apply Banach's fixed point theorem on each $\mathcal{Y}_j$, $0 \leq j \leq k-1$, and since $\mathcal{Y}_j \subset \mathcal{Y}$ for $0 \leq j \leq k$ the solutions are all equal to $u(\nu)$. Under assumptions (i)–(iii) we can formally compute the first $(k-1)$ derivatives of $u(\nu)$ considered as lying in $\mathcal{Y}$, if we assume hypotheses (i)–(iv) then we can even compute the formal k-th derivative of $u(\nu)$ considered as lying in $\mathcal{Y}$. It remains to be shown that the formal derivatives are indeed the derivatives of $u(\nu)$. For $k = 1$ the proof can be found in [31]. The rest is induction over k. Since this theorem is the main technical tool of our results we present the whole proof of the theorem.

1. Step. We first show that the solution $u(\nu)$ is a C^1-function of ν with respect to the norm of $\mathcal{Y}$. The formal derivative $\kappa(\nu)$ is given by the equation

$$\kappa(\nu) - (\partial_u \Pi)(u(\nu), \nu)\kappa(\nu) = (\partial_\nu \Pi)(u, \nu)|_{u=u(\nu)}.$$

Since $\|(\partial_u \Pi)(u(\nu), \nu)\|_{\mathcal{L}(\mathcal{Y})} \leq c < 1$ this equation can be solved uniquely for $\kappa(\nu) \in \mathcal{Y}$. Furthermore, $\kappa(\nu) \in \mathcal{Y}$ depends continuously on ν. We consider a fixed ν. In order to prove that $\kappa(\nu) = \partial_\nu u(\nu)$ we have to show that

$$\|u(\nu + \tilde{\nu}) - u(\nu) - \kappa(\nu)\tilde{\nu}\|_{\mathcal{Y}} = o(\tilde{\nu}). \tag{4.12}$$

Multiplying

$$\begin{aligned} u(\nu + \tilde{\nu}) - u(\nu) - \kappa(\nu)\tilde{\nu} &= u(\nu + \tilde{\nu}) - u(\nu) \\ &\quad -\tilde{\nu}(1 - (\partial_u \Pi)(u(\nu), \nu))^{-1}(\partial_\nu \Pi)(u, \nu)|_{u=u(\nu)} \end{aligned}$$

by $(1 - (\partial_u \Pi)(u(\nu), \nu))$ we see that (4.12) is equivalent to $\|\theta(\tilde{u}, \tilde{\nu})\|_{\mathcal{Y}} = o(\tilde{\nu})$ where $\tilde{u} = u(\nu + \tilde{\nu}) - u(\nu)$ and

$$\theta(\tilde{u}, \tilde{\nu}) = \Pi(u(\nu+\tilde{\nu}), \nu+\tilde{\nu}) - \Pi(u(\nu), \nu) - (\partial_u \Pi)(u(\nu), \nu)\tilde{u} - (\partial_\nu \Pi)(u(\nu), \nu)\tilde{\nu}.$$

We can estimate

$$\begin{aligned} \|\theta(\tilde{u}, \tilde{\nu})\|_{\mathcal{Y}} &\leq \|\Pi(u(\nu + \tilde{\nu}), \nu + \tilde{\nu}) - \Pi(u(\nu + \tilde{\nu}), \nu) - (\partial_\nu \Pi)(u(\nu), \nu)\tilde{\nu}\|_{\mathcal{Y}} \\ &\quad +\|\Pi(u(\nu + \tilde{\nu}), \nu) - \Pi(u(\nu), \nu) - (\partial_u \Pi)(u(\nu), \nu)\tilde{u}\|_{\mathcal{Y}} \\ &= \|(\partial_\nu \Pi)(u(\nu + \tilde{\nu}), \nu)\tilde{\nu} - (\partial_\nu \Pi)(u(\nu), \nu)\tilde{\nu}\|_{\mathcal{Y}} \\ &\quad +o(\|\tilde{u}\|_{\mathcal{Y}}) + o(\tilde{\nu}) \\ &\leq o(\tilde{\nu}) + o(\|\tilde{u}\|_{\mathcal{Y}}). \end{aligned} \tag{4.13}$$

Here we used that $u(\nu)$ is a C^0-function of ν with respect to the norm $\|\cdot\|_{\mathcal{Y}_1}$. This follows from Banach's contraction mapping theorem applied onto $\mathcal{Y}_1$ or, if $k = 1$, from the additional assumption (iv). It holds

$$\theta(\tilde{u}, \tilde{\nu}) = (1 - (\partial_u \Pi)(u(\nu), \nu))\tilde{u} - (\partial_\nu \Pi)(u(\nu), \nu)\tilde{\nu}.$$

Hence

$$\|\tilde{u}\|_{\mathcal{Y}} \leq \frac{1}{1-c}(\|(\partial_\nu \Pi)(u(\nu), \nu)\|_{\mathcal{Y}}|\tilde{\nu}| + \|\theta(\tilde{u}, \tilde{\nu})\|_{\mathcal{Y}}) \leq \frac{1}{1-c}(\tilde{c}|\tilde{\nu}| + o(\|\tilde{u}\|_{\mathcal{Y}})).$$

Thus, we obtain $\|\tilde{u}\|_{\mathcal{Y}} \leq \hat{c}|\tilde{\nu}|$ for $|\tilde{\nu}|$ small. From (4.13) we conclude that $\|\theta(\tilde{u}, \tilde{\nu})\|_{\mathcal{Y}} = o(\tilde{\nu})$. Hence $u(\nu)$ is a C^1-function of ν with respect to the norm $\|\cdot\|_{\mathcal{Y}}$.

2. Step. To show that $u(\nu)$ is a C^i-function of ν, $i > 1$, we proceed by induction. If the theorem holds for $k = (j-1)$, $j \geq 2$, and Π satisfies assumptions (i) –(iii) of the theorem with $k = j$ then by the contraction principle applied on $\mathcal{Y}_{j-1}$ the function $u(\nu)$ is continuous in ν with respect to the norm $\|\cdot\|_{\mathcal{Y}_{j-1}}$. Hence by part b) of the theorem for $k = (j-1)$ we conclude that $u(\nu)$ is C^{j-1}-smooth in ν when considered as lying in $\mathcal{Y}$. This proves part a) of the theorem for $k = j$. Now we come to part b). If Π satisfies assumptions (i)–(iv) of the theorem for $k = j$ then $u(\nu)$ is a C^{j-1}-function of ν in the $\mathcal{Y}$-norm and $u(\nu)$ is a $C^{j-\ell}$-function of ν with respect to the norm $\|\cdot\|_{\mathcal{Y}_\ell}$, $j \geq \ell \geq 1$. Therefore we can apply the theorem with $k = (j-1)$ onto the differentiated equation

$$(1 - (\partial_u \Pi)(u(\nu), \nu))\partial_\nu u(\nu) = (\partial_\nu \Pi)(u, \nu)|_{u=u(\nu)}.$$

and conclude that $\partial_\nu u(\nu)$ is a C^{j-1}-function of ν and that u is a C^j-function of ν with respect to the norm $\|\cdot\|_{\mathcal{Y}}$. ∎

4.5 Proof of the theorems on periodic forcing

We prove Theorem 2.3 by applying Theorem 4.8 onto (4.2) with $\nu = (g, q, \omega_{\text{ext}}, \mu)$ and with the hierarchy $\mathcal{Y}_j$ of Banach spaces defined by

$$\mathcal{Y}_j = (1 - \hat{P})Y_j, \quad Y_j \text{ given by (4.4)}, \quad 0 \leq j \leq k.$$

As before $\hat{P}$ is a projection which is $\mathcal{L}(Y, Y_{k-1})$-near the spectral projection P onto the center-unstable eigenspace and such that $\hat{P}\rho_g$ and $\rho_g \hat{P}$ are C^k-smooth in g in the $\mathcal{L}(Y)$-norm. We consider the fixed point equation $y = \Pi^\ell(y)$ with ℓ so large that $\mathrm{D}\Pi^\ell$ is a contraction on each Y_j, $0 \leq j \leq k-1$, $k \in \mathbb{N}$. Because of Lemma 4.7 all assumptions of Theorem 4.8 are satisfied. So there is a locally unique solution $y(q, g, \omega_{\text{ext}}, \mu) \in Y_k$ of $\Pi^\ell(y) = y$ if $(q, g, \omega_{\text{ext}}, \mu)$ is near $(q^*, g^*, \omega_{\text{ext}}^*, \mu^*)$ satisfying $y(q^*, g^*, \omega_{\text{ext}}^*, \mu^*) = y^*$ and $y(q, g, \omega_{\text{ext}}, \mu)$ is a C^{k-j}-function of $(q, g, \omega_{\text{ext}}, \mu)$ with respect to the norm $\|\cdot\|_{Y_j}$, $0 \leq j \leq k$. Since with $y = y(q, g, \omega_{\text{ext}}, \mu)$ also $\Pi^i(y)$, $i \in \mathbb{Z}$, are solutions of $\Pi^\ell(y) = y$ and the solution is locally unique, $y(q, g, \omega_{\text{ext}}, \mu)$ is also a solution of (4.2).

The reduced equation (4.3) is C^{k-j}-smooth in its variables if $y(q, g, \omega_{\text{ext}}, \mu)$ is considered as lying in Y_j. Solving the reduced equation by the ordinary implicit function theorem we obtain relative periodic orbits $\Phi_{T_{\text{ext}}}(u(\omega_{\text{ext}}, \mu)) = \rho_{g(\omega_{\text{ext}}, \mu)} u(\omega_{\text{ext}}, \mu)$ of (1.3) to the parameters ω_{ext}, μ with $(\omega_{\text{ext}}, \mu)$ near

$(\omega_{\text{ext}}^*, \mu^*)$. Here $g(\omega_{\text{ext}}, \mu)$ is C^k-smooth in $(\omega_{\text{ext}}, \mu)$ and $u(\omega_{\text{ext}}, \mu)$ depends C^{k-j}-smoothly on $(\omega_{\text{ext}}, \mu)$ when considered as lying in Y_j. ∎

Proposition 2.17 is proved along the same lines.

4.6 Proof of the results on Hopf bifurcation by use of Lyapunov-Schmidt-reduction

In this section we will prove Theorem 3.2 and Proposition 3.7 by Lyapunov-Schmidt-reduction.

4.6.1 Proof of Theorem 3.2

We will study the solutions of the equation

$$0 = F(u, g, T, \mu) := \begin{pmatrix} \rho_g^{-1}\Phi_T(u, \mu) - u \\ l_i(u - u^*(\mu)),\ i = 1, \dots, m \end{pmatrix}, \tag{4.14}$$

where $l_i \in Y^\star$ and the conditions $l_i(u - u^*) = 0$, $i = 1, \dots, m$, define a section transversely to the G-orbit of u^*. Later on, we will need an additional condition to take care of the time-shift symmetry of the relative periodic orbits which we want to find. The map F is smooth in u, μ, T for $T > 0$, but only continuous in g. Further $\partial_u F(u, g, T, \mu)$ is not continuous in g with respect to the norm $\|\cdot\|_{\mathcal{L}(Y)}$. So we can not use the usual Lyapunov-Schmidt-reduction to solve equation (4.14), but (4.14) fits into the setting which we treated in the preceding subsections, and we will use the techniques developed in these subsections to solve (4.14).

Let P be the projection onto the center-unstable eigenspace of u^*. By Lemma 4.7 we have $P \in \mathcal{L}(Y, Y_{k-1})$ and $P^\star \in \mathcal{L}(Y^\star, Z_{k-1}^\star)$ with the hierarchy of Banach spaces $\{Y_j\}_{0 \le j \le k}$ defined by (4.4). We can find a projector $\hat{P}$ which is near P in the norm of $\mathcal{L}(Y, Y_{k-1})$ and such that $\rho_g \hat{P}$ and $\hat{P}\rho_g$ are C^k-smooth in g in the norm of $\mathcal{L}(Y)$. Consider the fixed point equation

$$y = \Pi(y, q, g, T, \mu) := (1 - \hat{P})\rho_g^{-1}\Phi_T(y + q, \mu),$$

with $y \in (1 - \hat{P})Y$, $q \in \hat{P}Y$, on the scale of Banach spaces Y_j, $0 \le j \le k$. This fixed point equation equals (4.2) from Section 4 with T_{ext} replaced by T and $\Phi_t(\cdot)$ autonomous. So we get a solution $y(q, g, T, \mu)$ of the fixed point equation which is C^{k-j}-smooth in its parameters in the Y_j-norm. Now we are ready to solve the reduced equation $F_{\rm red}(q,g,T,\mu)=0$ with $F_{\rm red}$ given by

$$F_{\rm red}(q,g,T,\mu)=\begin{pmatrix} P\left(\rho_g^{-1}\Phi_T(q+y(q,g,T,\mu),\mu)-q-y(q,g,T,\mu)\right) \\ l_i(u-u^*), \quad i=1,\dots,m\end{pmatrix}. \tag{4.15}$$

The map $F_{\rm red}$ is C^{k-j}-smooth in its variables when considered as map from Y_j into Y_j. The rest of the proof is standard, see [6]:

Let $u^*(\mu)$ be a path of relative equilibria such that $l_i(u^*-u^*(\mu))=0$, $i=1,\dots,m$. Let $\Psi_t(\cdot)$ be the semiflow in a comoving frame

$$\Psi_t(u;\mu)=\rho_{g(\Phi_t(u,\mu))}\Phi_t(u+u^*(\mu);\mu)-u^*(\mu)$$

where u is Y_1-near u^*, $l_i(u-u^*)=0$, $i=1,\dots,m$, and $g(u)$ is such that $l_i(\rho_{g(u)}u-u^*)=0$, $i=1,\dots,m$, see also (3.4). Let $\mathrm{D}\Psi_t(0;0)=\mathrm{e}^{\tilde{L}t}$ and denote by P_ℓ the projection onto the space $l_i(u)=0$, $i=1,\dots,m$ such that $P_\ell \operatorname{alg}(G)u^*=0$. Then $\tilde{L}=P_\ell L^* P_\ell$.

Let w be the eigenvector of $\tilde{L}$ to the eigenvalue i. Let $\langle w^\star,\cdot\rangle$ belong to the left eigenspace of L^* to the eigenvalue i such that

$$\begin{aligned} \langle \operatorname{Re} w^\star, \operatorname{Re} w\rangle &= \langle \operatorname{Im} w^\star, \operatorname{Im} w\rangle = 1,\\ \langle \operatorname{Re} w^\star, \operatorname{Im} w\rangle &= \langle \operatorname{Im} w^\star, \operatorname{Re} w\rangle = 0 \end{aligned} \tag{4.16}$$

is satisfied and $\langle w^\star, \operatorname{alg} Gu^*\rangle=0$, $i=1\dots m$.

Let

$$s(q,g,T,\mu):=\frac{1}{2\pi}\langle \operatorname{Re} w^\star, \int_0^{2\pi} \mathrm{e}^{\tilde{L}(2\pi-t)}\Psi_{\frac{Tt}{2\pi}}(y(q,g,T,\mu)+q-u^*(\mu);\mu)\mathrm{d}t\rangle.$$

We first compute $q=q(s,T,\mu)$ and $g=g(s,T,\mu)$ as functions of s, T and μ by solving

$$F_{\rm red}-\langle \operatorname{Re} w^\star, F_{\rm red}\rangle \operatorname{Re} w-\langle \operatorname{Im} w^\star, F_{\rm red}\rangle \operatorname{Im} w=0.$$

and

$$\langle \operatorname{Im} w^\star, \int_0^{2\pi} \mathrm{e}^{\tilde{L}(2\pi-t)}\Psi_{\frac{Tt}{2\pi}}(y(q,g,T,\mu)+q-u^*(\mu);\mu)\mathrm{d}t\rangle=0.$$

The last condition fixes the time-shift. Now we still have to solve the C^k-function $\hat{F}:\mathbb{R}^3\to\mathbb{R}^2$, $(s,T,\mu)\to\hat{F}(s,T,\mu)$ given by

$$\hat{F}(s,T,\mu)=\begin{pmatrix}\langle \operatorname{Im} w^\star, F_{\rm red}(q(s,T,\mu),g(s,T,\mu),T,\mu)\rangle\\ \langle \operatorname{Re} w^\star, F_{\rm red}(q(s,T,\mu),g(s,T,\mu),T,\mu)\rangle\end{pmatrix}=0.$$

Obviously $\hat{F}(0,T,\mu)=0$, $\partial_{(s,T,\mu)}\hat{F}(0,2\pi,0)=0$. We define

$$F_{LS}(s,T,\mu) := \frac{1}{s}\hat{F}(s,T,\mu).$$

Since $\lim_{s\to 0}\frac{1}{s}\hat{F}(s,T,\mu) = \partial_s F_{LS}(0,T,\mu)$ and $\partial_s\hat{F}(0,2\pi,0) = 0$ we have $F_{LS}(0,2\pi,0) = 0$. Furthermore $\partial_T F_{LS}(0,T,\mu) = \partial_T\partial_s\hat{F}(0,T,\mu)$ and $\partial_\mu F_{LS}(0,T,\mu) = \partial_\mu\partial_s\hat{F}(0,T,\mu)$, since $\partial_{(T,\mu)}\hat{F}(0,T,\mu)=0$.

Lemma 4.9 *Under the assumptions of Theorem 3.2 the derivative*

$$\partial_{(T,\mu)}F_{LS}(0,2\pi,0)$$

of F_{LS} in $(s,T,\mu)=(0,2\pi,0)$ has full rank.

Proof of Lemma 4.9. We have

$$\partial_{(T,\mu)}F_{LS}(0,2\pi,0) = \begin{pmatrix} \langle \operatorname{Im} w^\star, \partial_{(T,\mu)}\partial_u\Psi_{2\pi}(0;0)\operatorname{Re} w\rangle \\ \langle \operatorname{Re} w^\star, \partial_{(T,\mu)}\partial_u\Psi_{2\pi}(0;0)\operatorname{Re} w\rangle \end{pmatrix} \tag{4.17}$$

We invoke the following lemma which is the adaption of a lemma in Crandall & Rabinowitz [6] to our setting.

Lemma 4.10 *Let assumptions (i)–(iii) of Theorem 3.2 hold. Then*

$$\begin{aligned} \langle \operatorname{Re} w^\star, \partial_\mu\partial_u\Psi_{2\pi}(0;0)\operatorname{Re} w\rangle &= 2\pi\operatorname{Re}\tfrac{\partial\beta}{\partial\mu}(0), \\ \langle \operatorname{Im} w^\star, \partial_\mu\partial_u\Psi_{2\pi}(0;0)\operatorname{Re} w\rangle &= -2\pi\operatorname{Im}\tfrac{\partial\beta}{\partial\mu}(0), \end{aligned} \tag{4.18}$$

where $\langle w^\star,\cdot\rangle$ is the left eigenvector of $\tilde{L}$ to the eigenvalue i *which satisfies (4.16).*

We have

$$\partial_T\partial_u\Psi_{2\pi}(0;0)\operatorname{Re} w = \tilde{L}e^{\tilde{L}2\pi}\operatorname{Re} w = \operatorname{Im} w.$$

Using Lemma 4.10 and condition (iv) we conclude that $\partial_{(T,\mu)}F_{LS}(0,2\pi,0)$ has full rank. ∎

Because of Lemma 4.9 we can apply the ordinary implicit function theorem to obtain solutions $u(s) := u^*(\mu(s)) + z(s)$, $g(s)$, $T(s)$, $\mu(s)$ of (4.14) which are relative periodic orbits with $l_i(z(s)) = 0$, $i=1,\dots,m$. Here $\mu(s)$, $g(s)$, $T(s)$ are C^{k-1}-smooth in s. Moreover, $z(s)$ is C^{k-1}-smooth in the $\|\cdot\|_Y$-norm and C^{k-j-1}-smooth in the $\|\cdot\|_{Y_j}$-norm, $1\le j\le k-1$.

We have $z(-s) = \Psi_{\frac{T(s)}{2}}(z(s);\mu(s))$. Since $\mu(s)$, $T(s)$ do not depend on the time-shift, they are even in s.

The solutions $u(s) = u^*(\mu(s)) + z(s)$, $g(s)$, $T(s)$, $\mu(s)$ of (4.14) which we obtained above depend C^{k-1}-smoothly on the chosen eigenvector of $\tilde{L}$ to the eigenvalue i. We can also consider $z \in Y$, g, T, μ as C^{k-1}-smooth functions of $(s_1 + \mathrm{i}s_2)w$ where $s_1, s_2 \in \mathbb{R}$ and w is the originally chosen eigenvector of $\tilde{L}$ to i. As before, $z(s_1, s_2)$ is C^{k-j-1}-smooth in the Y_j-norm, $1 \leq j < k$. We have $s = (s_1, 0)$ and

$$\mathrm{e}^{\mathrm{i}t} = \langle \mathrm{Re}\, w^\star + \mathrm{i}\,\mathrm{Im}\, w^\star, \int_0^{2\pi} \mathrm{e}^{\tilde{L}(2\pi - t_0)} \Psi_{\frac{(t_0+t)T(s)}{2\pi}}(z(s);\mu(s))\mathrm{d}t_0 \rangle.$$

Obviously $\mu(s_1, s_2)$, $T(s_1, s_2)$ only depend on $|(s_1, s_2)|$, and

$$z(s_1, s_2) := \Psi_{\frac{tT(s)}{2\pi}}(z(s);\mu(s)), \quad \text{where} \quad s_1 = s\cos t, \quad s_2 = s\sin t.$$

■

4.6.2 Equivariant Hopf bifurcation

In this subsection we prove Lemma 3.7, see also section 3.3. If the isotropy K of the relative equilibrium u^* is non-trivial it may happen that forced by symmetry the eigenspace of the K-equivariant matrix $\tilde{L}$ to the Hopf eigenvalue i has dimension higher than 2. Then the assumptions of Theorem 3.2 are not satisfied any more.

We choose functionals l_i, $i = 1, \ldots, \tilde{m}$, which define a section transversal to the group orbit Gu^* in u^* such that the transversal section $\{l_i(u - u^*) = 0, \quad i = 1, \ldots, \tilde{m}\}$ is K-invariant and that P_l is K-equivariant, see subsection 3.3. Here $\tilde{m} = \dim(G/K)$.

If $\Theta(K) = Z_\ell$ then we solve the equation

$$F(u, g, T, \mu) = \begin{pmatrix} \rho(gh^*)^{-1}\Phi_T(u,\mu) - u \\ l_i(u - u^*), \quad i = 1, \ldots \tilde{m} \end{pmatrix} = 0 \tag{4.19}$$

on $\mathrm{Fix}(\tilde{K})$ where $T \approx 2\pi/\ell$, $g \approx \mathrm{e}^{\frac{2\pi}{\ell}\xi^*}$, the group H is generated by $h^* \in K$ and $\tilde{K} = \ker(\Theta)$ is axial. By our assumptions $\mathrm{D}F_u(u^*, \mathrm{e}^{\frac{2\pi}{\ell}\xi^*}, 2\pi/\ell, 0)$ has a two-dimensional kernel and can thus be solved by the methods of subsection 4.6.1.

If $\Theta(K) = S^1$ we solve

$$F(u, g, T, \mu) = \begin{pmatrix} \rho(g\mathrm{e}^{-\chi^* T})^{-1}\Phi_T(u, \mu) - u \\ l_i(u - u^*), \quad i = 1, \dots \tilde{m} \end{pmatrix} = 0 \tag{4.20}$$

on $\mathrm{Fix}(\tilde{K})$. In this case the axial group H is generated by $\chi^* \in \mathrm{alg}(K)$ and $\tilde{K} = \ker(\Theta)$. We choose T such that $\rho_{\mathrm{e}^{-\chi^* T}}\mathrm{e}^{\tilde{L}T} - 1$ has a two-dimensional kernel. This is possible because the number of center eigenvalues of $\tilde{L}$ is finite. Then we can proceed as before. ∎

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