From asymptotics to exact results in Physics and Mathematics

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11th SEMPS, University of Surrey, March 28, 2018
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2 Setting up Resurgence

3 Applications
   • Summation: Painlevé I and Matrix Models
   • Interpolation: Cusp Anomalous Dimension
   • Prediction: QNM in $\mathcal{N} = 4$ SYM

4 Summary/Future Directions
Perturbation theory: fundamental in computations of
- energies in quantum mechanics
- Solutions on NLODEs
- beta-functions in quantum field theory
- genus expansions of string theory
- large $N$ expansion of non-abelian gauge theories

... 

Goal: to understand analytic properties beyond numerical computations

BUT... most perturbative expansions are asymptotic, i.e. zero radius of convergence!

- Why? due to non-perturbative "semi-classical" effects such as
  - instantons
  - renormalons
  - Other objects not captured by a perturbative analysis
Double Well in Quantum Mechanics

\[
V(x) = \frac{x^2}{2} (1 + \sqrt{g} x)^2
\]

e.g. \( V(x) = \frac{x^2}{2} (1 + \sqrt{g} x)^2 \)

\[\text{Hamiltonian}\]
\[
H = -\frac{1}{2} \left( \frac{d}{dx} \right)^2 + V(x)
\]

\[\text{Schrödinger eq}\]
\[
H\psi(x, g) = E(g)\psi(x, g)
\]

\[g = 0 \Rightarrow \text{Harmonic oscillator}\]
\[
V_H(x) = \frac{1}{2} x^2
\]
\[
E_{g.s.} = \frac{1}{2}
\]

\[g > 0 \text{ How can we solve it?}\]
Double Well in Quantum Mechanics

Take coupling $g$ very small
Double Well in Quantum Mechanics

Take coupling $g$ very small

Ground-state energy:

$$E_{g.s.}(g) \simeq \sum_{n=0}^{\infty} E_n g^n$$

where $E_0 = 1/2$

Questions:

- Does the series converge? No! **Asymptotic series**
- Exact results? **Borel transform & resummation**
Aside: Asymptotic series

\[ f(g) \simeq \sum_{n \geq 0} f_n g^n \]

- Divergent! No matter how small \( g \) is: \( f_n g^n \to \infty \)
- Truncate at some optimal \( n = N \): very good approximation
- Take \( g \ll 1 \) fixed: define truncation \( f_N(g) = \sum_{n=0}^{N} f_n g^n \)

\[ \log(f - f_N) \]

Non-perturbative effect: \( g \to 0 \) invisible in perturbation theory!

\[ e^{-A/g} \]
Aside: Asymptotic series

\[ f(g) \approx \sum_{n \geq 0} f_n g^n \]

- Divergent! No matter how small \( g \) is: \( f_n g^n \to \infty \)
- Truncate at some optimal \( n = N \): very good approximation
- Take \( g \ll 1 \) fixed: define truncation \( f_N(g) = \sum_{n=0}^{N} f_n g^n \)

Double-well Potential: \( E_n \sim n! A^{-n} \) instantons!

If we fix \( E_n \sim n! A^{-n} \), if we fix

\[ g = \frac{1}{100} \]

\[ N_{\text{op}} = \frac{A}{g} \approx 16 \]

Optimal error:

\[ (f - f_N)(g) \sim e^{-A/g} \approx 5.7 \times 10^{-8} \]
Analytic properties? Resummation

Perturbative expansion of quantity $F(g)$ in parameter $g \sim 0$

$$F(g) \sim \sum_{n \geq 0} F_n g^{n+1}, \quad \text{Asymptotic series: } F_n \sim n!$$

▶ How to find $F(g)$?

▶ Borel transform $\mathcal{B}[F]$: "remove" the factorial growth
▶ Analytically continue $\mathcal{B}[F]$ to full complex plane
▶ Define resummation $SF$ by the inverse Borel transform
Aside: Borel Transform & Resummation

Asymptotic series: \( F(g) \cong \sum_{n \geq 0} F_n g^{n+1} \), with \( F_n \sim n! \)

- **Borel transform:**
  \[ \mathcal{B}[F](s) = \sum_{n=0}^{\infty} \frac{F_n}{n!} s^n \]
  Rule: \( \mathcal{B}[g^{\alpha+1}](s) = s^\alpha / \Gamma(\alpha + 1) \)
  - finite radius of convergence - find function \( \mathcal{B}[F](s) \)
  - In general \( \mathcal{B}[F](s) \) will have singularities

- **Borel resummation** of \( F \) is the Laplace transform
  \[ SF(g) = \int_{0}^{\infty} ds \mathcal{B}[F](s)e^{-s/g} \]
\[ F(g) \simeq \sum_{n \geq 0} F_n g^{n+1}, \quad \text{Asymptotic series: } F_n \sim n! \]

Borel resummation of \( F \) along direction \( \theta \) is the Laplace transform

\[ S_\theta F(g) = \int_0^{e^{i\theta} \infty} ds \, B[F](s)e^{-s/g} \]

▶ **BUT:** \( SF \) is just a Laplace transform - needs an integration contour to be properly defined!

▶ **If we have a singularity in the complex Borel plane:**

**Nonperturbative ambiguity:** ambiguity in choosing how integration contour will avoid the singularity.
Nonperturbative Ambiguity

Borel resummation of $F$ along direction $\theta$ is the Laplace transform

$$S_\theta F(g) = \int_0^{e^{i\theta}\infty} ds \, \mathcal{B}[F](s)e^{-s/g}$$

- Take $\mathcal{B}[F](s)$ with singularities in direction $\theta$:

  Nonperturbative ambiguity:

  $\mathcal{B}[F](s) \sim \frac{1}{s-A}$ in direction $\theta$

  $$S_+ F(g) - S_- F(g) \sim \exp\left(-\frac{A}{g}\right)$$

  around $g \sim 0$ this is non-analytic

- Singularities in the Borel plane occur along Stokes lines

  Perturbative series is non-Borel resummable along Stokes lines
Glimpse into Resurgence

- Borel plane singularities:
  - Related to non-perturbative data
  - Govern asymptotic behaviour of original perturbative series

Non-perturbative information **resurges** in the perturbative data!

- Understanding the resurgent properties of our solution:

Obtain a non-ambiguous, global, analytic result
How can we achieve this?
Beyond Perturbation Theory?

Learn from the example of anharmonic potential in QM
[Vainshtein’64, Bender,Wu’73]

- Perturbative series of ground-state energy:

\[ E^{(0)}(g) = \sum E^{(0)}_k g^k, \quad E^{(0)}_k \sim k! A^{-k}, \quad k \gg 1 \]

- Resummation along real axis: singularities and ambiguity!

What happens if we try to include instanton sectors?

- Expanding around each fixed instanton sector

\[ n - \text{instanton sector: } E^{(n)}(g) = e^{-nA/g} \sum E^{(n)}_k g^k \]

Also asymptotic, with large-order behaviour

\[ E^{(n)}_k \sim k! (nA)^{-k}, \quad k \gg 1 \]

All multi-instanton series suffer from nonperturbative ambiguities!
Infinite instanton sectors with nonperturbative ambiguities!  

Seems to make the problem with perturbation theory even worse!

**BUT:** for the ground state energy of double-well potential  
[Bogomolny, Zinn-Justin ’80-83]

- ambiguity in 2-instanton sector *precisely* cancels ambiguity in perturbative expansion
- ambiguity in 3-instanton sector cancels ambiguity in 1-instanton sector
- ...

**Multi-instantonic ambiguities** are the *solution* to our problem!
Beyond Perturbation Theory!

**Ground-state energy** = sum over all multi-instanton sectors

Ambiguities arising in different sectors conspire to cancel each other
The final result is *real* and *free* from any nonperturbative ambiguities!

How to implement this sum? **Transseries ansatz!**

**Transseries**: formal power series in two or more variables, each a function of the parameter $z \sim 0$

$$E(g, \sigma) = \sum_{n \geq 0} \sigma^n E^{(n)}(g), \quad E^{(n)}(g) \simeq e^{-nA/g} \sum_{k \geq 1} E_k^{(n)} g^k$$

▶ our case has $e^{-A/g}$ and $g$
▶ $\sigma$: instanton counting parameter
Ambiguities along Stokes lines

\[ E(g, \sigma) = \sum_{n \geq 0} \sigma^n E^{(n)}(g), \quad E^{(n)}(g) \simeq e^{-nA/g} \sum_{k \geq 1} E_k^{(n)} g^k \]

- If \( B[E^{(n)}] \) has singularities in a direction \( \theta \) (Stokes line)
  - \( E^{(n)}(g) \) has an associated ambiguity: \( (S_{\theta^+} - S_{\theta^-}) E^{(n)} \neq 0 \)

- **BUT**: \( S_{\theta \pm} E \) are related:

  \[ S_{\theta^+} E^{(n)} = S_{\theta^-} \circ \left( E^{(n)} - \text{Disc}_{\theta} E^{(n)} \right) \]

  - \( \text{Disc}_{\theta} \neq 0 \) encodes Stokes transition at \( \theta \)

- Cancelling ambiguities:
  - Choose \( \sigma = \sigma_0 \) such that \( (S_{\theta^+} - S_{\theta^-}) E(z, \sigma_0) = 0 \)
  - Non-ambiguous result is \( \frac{1}{2} (S_{\theta^+} + S_{\theta^-}) E(z, \sigma_0) \)

Calculating Ambiguities and Discontinuities? Via **Resurgence**
Cancelation of ambiguities in multi-instanton sectors: larger structure behind perturbation theory!

**Resurgence analysis and Transseries**

A transseries \( z = \frac{1}{g} \sim \infty \)

\[
F(z, \sigma) = \sum_{n \geq 0} \sigma^n F^{(n)}(z), \quad F^{(n)}(z) \sim e^{-nAz} \sum_{k \geq 0} F^{(n)}_k z^{-k}
\]

defines a resurgent function if it relates the asymptotics of multi-instanton contributions \( F^{(\ell)}_n \) in terms of \( F^{(\ell')}_n \) where \( \ell' \) is close to \( \ell \)

**How does it work?**
Multi-instanton asymptotic series

\[ F(z) = \sum_{n=0}^{\infty} \sigma^n F^{(n)}(z) \]

Perturbative series:  \( F^{(0)}(z) = \sum_{g=0}^{\infty} F_g^{(0)} z^{-g-1} \)

Instanton series:  \( F^{(n)}(z) = e^{-nA}z \sum_{g=1}^{\infty} F_g^{(n)} z^{-g} \)
Large-order behaviour - Perturbative series for large $g$

$$F_g^{(0)} \sim S_1 \sum_{n>0} a_n(g) F_n^{(1)} + 2^{-g} S_1^2 \sum_{n>0} b_n(g) F_n^{(2)} + \ldots$$

All multi-instanton sectors contribute to the large-order behavior of coefficients $F_g^{(0)}$
Equivalently: Perturbative series for large $g$ ENCODES all other sectors

\[ F_g^{(0)} \sim S_1 F_1^{(1)} + \ldots \]

From the leading large $g$ behaviour of $F_g^{(0)}$:

determine $F_1^{(1)}, F_2^{(1)}, \ldots$
Equivalently: Perturbative series for large $g$ ENCODES all other sectors.

\[ F_g^{(0)} - S_1 \sum_{n>0} a_n(g) F_n^{(1)} \sim 2^{-g} S_1^2 \left( F_1^{(2)} + \frac{2A}{g-1} F_2^{(2)} + \ldots \right) + O(3^{-g}) \]
Resummation and analytic results

Full solution defined by transseries ($z \sim \infty$)

$$F(z, \sigma) = \sum_{n \geq 0} \sigma^n e^{-nAz} \Phi^{(n)}(z), \quad \Phi^{(n)}(z) \sim z^\beta n \sum_{k \geq 0} F^{(n)}_k z^{-k}$$

How to evaluate it? Depends on the value of $z \in \mathbb{C}$

- If $\text{Re}(Az) > 0$, non-perturbative sectors exponentially suppressed:
  
  Borel summation

  $$S_\theta F(z, \sigma) = S\Phi^{(0)}(z) + \sigma e^{-Az} S\Phi^{(1)}(z) + \mathcal{O}(e^{-2Az})$$

  we can obtain results for large AND small couling ($z \ll 1$)

- If $\text{Re}(Az) = 0$, all sectors of the same order:
  
  Analytic transseries summation

  $$S F(z, \sigma) = \sum_{n \geq 0} \sigma^n e^{-nAz} z^\beta n F^{(n)}_0 + \frac{1}{z} \sum_{n \geq 0} \sigma^n e^{-nAz} z^\beta n F^{(n)}_1 + \mathcal{O}(z^{-2})$$

  we can obtain analytic information, e.g. zeros of the solution
Applications!
Resurgence in Quantum Theories

- Many recent applications of resurgence
  - Ordinary integrals and non-linear differential equations
  - Quantum Mechanics: Exact WKB, ambiguity cancelations
  - QFTs: fractional instantons, UV renormalons, OPEs
  - Matrix models: generalised instanton sectors
  - String theory: holomorphic anomaly equation

Next:

1. **Analytic summation** [Garoufalidis, Its, Kapev, Mariño, IA, Schiappa, Vaz, Vonk, ’10 - ’18]
   - Global solutions of NLODEs: Painlevé I
   - Asymptotics of matrix models at large $N$

2. **Ambiguity cancelation and interpolation** [IA, ’15]
   - Cusp anomalous dimension at large coupling

3. **Prediction of nonperturbative phenomena** [IA, Spaliński’15, on-going]
   - Quasi-normal modes in $\mathcal{N} = 4$ SYM
Summation and analytic results

Painlevé I and Matrix Models
Painlevé I, 2d Gravity and Matrix models

- Matrix models:
  - NP description of string theory in simpler backgrounds: non-critical strings and Dijkgraaf-Vafa type topological strings [Dijkgraaf, Vafa '02]
  - Simper models for studying NP structure behind large $N$ 't Hooft expansions
  - Can help us understand large-$N$ duality

- 2d quantum gravity is obtained by taking a double scaling limit: large $N$ and small coupling $g_s$ [Douglas, Shenker '90][Brézin, Kazakov '90][Gross, Migdal '90]

- Free energy of 2d gravity related to the Painlevé I NLODE
  \[ u^2 - \frac{1}{6} u'' = z \]
  - $u(z) = -F''(z)$ where $z^{-5/4} \sim g_s$.

- Study Painlevé I: simpler model, already showing major features from string theory
  - Asymptotic series with $(2g)!$ growth $\Rightarrow g_s^2$ expansion
Use a **2-parameter transseries**: [Garoufalidis, Its, Kapaev, Mariño ’10] [IA, Schiappa, Vonk ’11]

\[
u(x; \sigma_1, \sigma_2) = \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sigma_1^n \sigma_2^m e^{-\frac{(n-m)A}{x}} \Phi(n|m)(x)
\]

- Two instanton actions \( A = \pm 8\sqrt{3}/5 \): evidence of resonance, many sectors with same exponential grading

- \( x = z^{-5/4} \sim g_s \) is open string coupling; \( \sigma_i \) are boundary data

- Asymptotic series: \( \Phi(n|m)(x) \) have a topological genus expansion \( (g_s^2) \), \( \Phi(n|m), n \neq m \) have expansions in \( g_s \): evidence of resonance

- **Sectorial solutions in Painlevé I:** specified by boundary data \( \sigma_i \)
  - Different \( \sigma_i \) determine different solutions and asymptotics
  - Stokes phenomena: "glue" different sectors to build global solutions
Painlevé I solutions

\[ u(x, \sigma) = \sum_{n \in \mathbb{N}_0^2} \sigma^n e^{-n \cdot A / x} \phi_n(x), \quad A \equiv (A, -A), \quad \sigma^n \equiv \sigma_1^n \sigma_2^m \]

0 parameter: Tritronquée

1 parameter: Tronquée

2 parameter

4 empty "quintants"

2 empty "quintants"

General

Can we "sum" the transseries into a function? Take \( \sigma_2 = 0 \)

- If \( e^{-A/x} \) is exp. suppressed: Borel-Padé summation
- If \( e^{-A/x} \sim 1 \): analytic transseries summation \( \Rightarrow \) analytical data

- Mathematical interpretation: anti-Stokes line
- Physical interpretation: phase transition
Painlevé I Partition function $\mathcal{Z}$

- Define Partition function: $\mathcal{Z}(x, \sigma) = e^F$ with $F'' \equiv u$
- Analytic transseries summation: Allows us to go inside the "filled sectors"

\[
\mathcal{Z}(x, \sigma) = \sum_{n=0}^{+\infty} \left( \sigma_1 e^{-A/x} \right)^n x^{\beta_n} F_0^{(n)} + x \sum_{n=0}^{+\infty} \left( \sigma_1 e^{-A/x} \right)^n x^{\beta_n} F_1^{(n)} + \cdots
\]

- Sectors with poles of $u$, zeros of $\mathcal{Z}$.
- Find locations of all zeros of the partition function from the transseries [Costin et al, '95-13; IA, Schiappa, Vonk, on-going]

\[
\mathcal{Z}_0 (\zeta, q) = \sum_{n=0}^{+\infty} G_2(n+1) \zeta^n q^{n^2}, \quad \zeta \sim \sigma_1 e^{-A/x}; \quad q \sim x^{1/2}
\]

- Only works for the adjoining sectors: to get to fifth sector: Stokes phenomena
Summation and analytic results

Large $N$ quartic matrix model
Quartic matrix model

Quartic model partition function \((N \times N\) matrix \(M\))

\[
Z(N, g_s) \propto \int dM \exp \left( -\frac{1}{g_s} \text{Tr} V(M) \right), \quad V(z) = \frac{1}{2}z^2 - \frac{1}{24} \lambda z^4
\]

Local solutions in "Stokes regions": saddle point analysis around 1-cut solution

Free energy has perturbative genus expansion at large \(N\)

\[
F \equiv \log Z \simeq \sum_{g \geq 0} F_g(t) g_s^{2g-2}, \quad t = g_s N
\]

- Obey a NP finite difference eq: string equation

\[
\mathcal{R}(t) \left( 1 - \frac{\lambda}{6} (\mathcal{R}(t-g_s) + \mathcal{R}(t) + \mathcal{R}(t+g_s)) \right) = t, \quad \mathcal{R}(n g_s) = r_n
\]

where \(r_n = \frac{Z_{n+1}Z_{n-1}}{Z_n^2}\) and \(\mathcal{R}(t)\) is directly related to the free energies
Quartic matrix model

\( \mathcal{R}(t) \) has **resurgent properties**:

\[
\mathcal{R}(t, \sigma_1, \sigma_2) = \sum_{n,m \geq 0} \sigma_1^n \sigma_2^m e^{-N(n-m)\frac{A(t)}{t}} t^{\beta_{nm}} R_{(n|m)}(t)
\]

- **\( R_{(n|m)}(t) \) asymptotic expansions**
- **Instanton action \( A(t) \) and coefficients \( R_{g_{(n|m)}}(t) \) are functions.**
- **Large-\( N \) phase diagram** (first studied in [Bertola ’07, Bertola, Tovbis ’11]): study the leading contributions to the exponentials:
  - **Stokes lines** \( \text{Im} \left( \frac{A(t)}{t} \right) = 0 \): instanton contributions maximally suppressed
  - **Anti-Stokes lines** \( \text{Re} \left( \frac{A(t)}{t} \right) = 0 \): all contributions of same order
- **Recover analytic data from the transseries**:
  - **Finite \( N \) results via Borel-Padé summation** [Couso-Santamaría, Schiappa, Vaz ’15]
  - **Lee-Yang zeros via analytic transseries summation** [IA, Schiappa, Vonk, on-going]
Phase Diagram

- **light blue:** Stokes regions, standard 't Hooft large $N$ expansion
  - I: 1-cut solution is dominant
  - II: 2-cut sym solution dominant

- **green:** anti-Stokes region, dominated by 3-cuts solution, modular properties; no genus expansion
  [Bonnet, David, Eynard '00]

- **light red:** trivalent tree-like configuration dominant

- Re line in I and II: Stokes lines, exponentially suppressed saddles are maximally suppressed

- P1 (P2): DS point described by Painlevé I (II) equation

Evidence of different phases?
What local solutions are associated with each phase?
How to obtain analytic data? Global Solutions?
The anti-Stokes phase: numerical evidence

- Numerically calculate the recursion coefficients $r_n$ with the boundary condition of the 1-cut configuration
- Take $N = 1000$ $\arg t = \frac{\pi}{12}$ fixed, change $|t|$ from the 1-cut phase into anti-Stokes
- $r$: normalization factor (classical solution $g_s = 0$)

Evidence of different phases: they lead to different asymptotics of the $R(t)$ in different regions
The anti-Stokes phase: numerical evidence

- Perform optimal truncation to the one-parameter sectors of $\mathcal{R}(t, \sigma_1, 0)$:
  - perturbative $R_{(0,0)}(t)$ plus $n$-instantons $R_{(n,0)}(t)$, for $n = 1, 2, 3$
- Compare to the numerical results for the $r_n$

Adding the first three instanton correction to the $\mathcal{R}(t)$, we cannot reach far into the anti-Stokes region: all instanton contributions are of the same order and need to be included.
Can we do better? Perform **analytic transseries summation**
Perform analytic transseries summation for the one-parameter partition function $Z(t) = e^F$

Sum the leading terms in $g_s$ for $Z(t)$

Determine the $R(t)$ from these results

Leading $g_s$ analytic transseries summation for $Z(t)$ follows the numerical results far into the anti-Stokes region!
Zeroes of the partition function

Use the analytic transseries summation to predict Lee-Yang zeros?

- **Left**: prediction of zeros of $Z(t)$ obtained from analytic transseries summation with $N = 10$ eigenvalues
- **Down**: numerical calculation of zeros from direct calculation of the matrix integral ($N = 100$). The grayscale is proportional to number of zeros

Leading $g_s$ quadratic transseries summation for $Z(t)$ predicts analytic results deep into the anti-Stokes region!
Cusp Anomalous Dimension
Cusp Anomalous Dimension

- Appears in $\mathcal{N} = 4$ SYM and strings in $AdS_5 \times S^5$

- Scaling behaviour of the anomalous dimension of a Wilson loop with a light-like cusp in the integration contour

\[ \langle W \rangle \sim e^{-\Gamma_{cusp} \log \frac{\Lambda_{UV}}{m_{IR}}} \]

- Scaling dimension of a twist-2 operator $\text{tr}(X^I D_{\mu_1} \cdots D_{\mu_5} X^I)$, at large spin $S$;

- Dispersion relation of long folded spinning strings in $AdS$:

\[ \Delta - S = f(g) \log S \]

$f(g)$: universal scaling function
From integrability it obeys the BES integral equations [Beisert, Eden, Staudacher, 07]

\[
\frac{\gamma(2gt)}{2gt} = K(2gt, 0) - 2g \int_0^\infty \frac{dt'}{e^{t'} - 1} K(2gt, 2gt') \gamma(2gt')
\]

\(K(t, t')\) is so-called BES Kernel [Eden, Staudacher, 06]

Cusp anomalous dimension given by

\[
\Gamma_{\text{cusp}}(g) = 8 \lim_{t \to 0} \frac{\gamma(2gt)}{2gt}.
\]

Weak coupling result \(g \ll 1\) known

Resurgent analysis: for \(g \gg 1\) expansion is asymptotic! [Basso, Korchemsky, Kotanski, 07]
Transseries and ambiguities [IA,15]

- Up to 2-instantons: 1-parameter transseries ansatz ($x = 8\pi g \gg 1$)

$$\frac{\Gamma_{\text{cusp}}(g, \sigma)}{2g} - 1 = \sum_{m=0}^{+\infty} \sigma^m e^{-mA_x} \Gamma^{(m)}(x); \quad \Gamma^{(m)}(x) \simeq x^{-m/2} \sum_{k=0}^{+\infty} \Gamma^{(m)}_k \left(\frac{x}{2}\right)^{-k}$$

- $\Gamma^{(m)}(x)$ are asymptotic series. Resurgent transseries? Yes!
  - sectors $\Gamma^{(0)}$, $\Gamma^{(1)}$ and $\Gamma^{(2)}$ related via large order relations

- $g$ real and positive: resummation of each sector $S_{\theta=0} \Gamma^{(m)}(x)$
  - But: $\theta = 0$ direction has singularities - it is a Stokes line!
  - We have an imaginary ambiguity: $(S_{0^+} - S_{0^-}) \Gamma^{(m)}(x) \neq 0$

- Use resurgence to cancel ambiguity: fix $\sigma_0 = \sigma_R + i \sigma_I$
  - $\Gamma_{\text{cusp}}(g, \sigma_0)$ no longer has imaginary part!

- Can we resum the transseries and obtain results for $g$ finite?

Yes: via the Borel-Padé resummation
Resummation and Results at Weak Coupling [IA,15]

- Resum the results up to 2nd nonperturbative order:

\[
\text{Resummation of Transseries } \text{Re}(\Gamma^{(0)}) + \sigma_R \text{Re}(\Gamma^{(1)}) + (\sigma_R^2 - \sigma_I^2) \text{Re}(\Gamma^{(2)})
\]

- Dashed – truncated sum of the perturbative expansion \( \Gamma^{(0)} \)
- Blue – known small coupling expansion (7 loops) of \( \Gamma(g) \)
Prediction of NP phenomena

Hydrodynamics
Hydrodynamic gradient expansion

- Evolution equations for energy-momentum tensor

\[ \nabla_\mu T^{\mu \nu} = 0 \]

- In hydrodynamic theories the E-M tensor is given by

\[ T^{\mu \nu} = \mathcal{E} u^\mu u^\nu + \mathcal{P}(\mathcal{E}) (\eta^{\mu \nu} + u^\mu u^\nu) + \Pi^{\mu \nu}, \]

- \( \mathcal{E} \) is energy density
- \( \Pi^{\mu \nu} \) is the shear stress tensor
- \( \mathcal{P}(\mathcal{E}) = \mathcal{E}/3 \) is pressure in \( d = 4 \) conformal theories
- \( u \) is flow velocity - timelike eigenvector of the E-M tensor

- Hydrodynamic gradient expansion: approximate \( \Pi^{\mu \nu} \) by series of corrections to ideal fluid behaviour
Hydrodynamic gradient expansion can be determined via the microscopic theory associated to the fluid.

For relativistic hydrodynamics with boost invariant flow: microscopic theory is large $N$ $\mathcal{N} = 4$ SYM at strong coupling.

Objective: determine energy density from gauge-gravity duality, by solving Einstein's equations with appropriate metric ansatz.

Non-hydrodynamic d.o.f. are exponentially decaying sectors of a transseries-type ansatz for the metric components, quasi-normal modes (QNM).

Determine perturbative part to very high order (240 terms)
[Heller, Janik, Witaszczyk, ’13]

Determine non-perturbative sectors to high order
[IA, Jankowski, Meiring, Spaliński, Witaszczyk, on-going]
Non-hydrodynamic modes and gradient expansion

Borel transform for the perturbative part of gradient expansion:

[Heller, Janik, Witaszczyk, ’13] [IA, Jankowski, Meiring, Spaliński, Witaszczyk, on-going]

QNM:

\[ \omega_1; 2\omega_1; 3\omega_1 \]

\[ \omega_2; \]

\[ \omega_3; \]

\[ \overline{\omega}_i; \]

\[ \omega_1 = \frac{3}{2} (2.746676 + 3.119452i); \]

\[ \omega_2 = \frac{3}{2} (4.763570 + 5.169521i); \]

\[ \omega_3 = \frac{3}{2} (6.769565 + 7.187931i); \]

Resurgence?
Transseries and NP predictions

- Multi-parameter transseries ansatz for the energy density

\[ \epsilon(\tau, \sigma) = \sum_n \sigma^n e^{-n \cdot A(\omega_i) \tau^{2/3}} \phi_n(\tau) \]

- Analyse the large order behaviour of the hydrodynamic series

\[ \phi_0(\tau) \approx \tau^{-4/3} \sum_{k=0}^{+\infty} \epsilon_k^{(0)} \tau^{-2k/3} \]

Convergence of \( \epsilon_k^{(0)} \) to first coefficients of \( \omega_1 \) sector

Convergence of resummed \( \epsilon_k^{(0)} \) to first coefficients of \( \omega_2 \) sector
Introduction to resurgence and applications to physical problems

- **Resurgence analysis:**
  - Transseries solutions
  - Predictions and large-order relations
  - Ambiguity cancelations
  - Summation and analytic results

- **Applications:**
  - Painlevé I NLODE and Large $N$ dynamics of matrix models
  - Strong coupling of cusp anomalous dimension
  - Strongly coupled fluid in $\mathcal{N} = 4$ SYM and gravitational QNM
Current work

- Analysis of phase diagram of quartic matrix model
  - Stokes transitions;
  - modular properties of the transseries

- Stokes transitions in Painlevé I

- Algebra structure of multi-parameter resurgent transseries, interplay between
  - coupling $g_{YM}, g_s \to 0$;
  - rank of gauge group $N \to \infty$;
  - ’t Hooft coupling $\lambda = g_{YM}^2 N$ fixed: large, small

- Applications of resurgence in string theory observables:
  - Bremsstrahlung function;
  - Lüscher corrections and the thermodynamic Bethe ansatz
Thank you!